# The algebraic structure of quaternionic analysis

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**Abstract**

The regularity of a quaternionic function is reinterpreted through a new canonical decomposition of the real differential, giving new insights into the algebraic properties of the regularity itself. The result comes from a somewhat unusual point of view on the automorphisms of the quaternionic field: a general notion of quaternionic linearity is associated to them, and some unnoticed metric properties of their inner representation are used to build up the theory.

## 1 Introduction

Quaternionic analysis is the theory of the quaternionic valued functions $f(q)$ of the quaternionic variable $q = t + ix + jy + kz$, which satisfy the Cauchy–Riemann type equation

$$\frac{\partial f}{\partial t} + i\,\frac{\partial f}{\partial x} + j\,\frac{\partial f}{\partial y} + k\,\frac{\partial f}{\partial z} = 0\ . \tag{1.1}$$

Since their introduction by Fueter ([6]) in 1935 , they are called *regular* functions, and the same name is reserved to the solutions of three other type of equations

$$\begin{aligned} &\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\,i + \frac{\partial f}{\partial y}\,j + \frac{\partial f}{\partial z}\,k = 0 \\ &\frac{\partial f}{\partial t} - i\,\frac{\partial f}{\partial x} - j\,\frac{\partial f}{\partial y} - k\,\frac{\partial f}{\partial z} = 0 \\ &\frac{\partial f}{\partial t} - \frac{\partial f}{\partial x}\,i - \frac{\partial f}{\partial y}\,j - \frac{\partial f}{\partial z}\,k = 0 \end{aligned} \tag{1.2}$$

Received by the editors November 2010.
Communicated by F. Brackx.
2000 *Mathematics Subject Classification* : 30G35, 32A30.
*Key words and phrases* : Quaternionic analysis; regular functions; quaternionic linearity.

which take into the account the noncommutativity of the quaternionic field $\mathbb{H}$ and the conjugate quaternionic variable $\overline{q} = t - ix - jy - kz$. All the classes have mirroring properties and, from the very beginning of the story, it is made clear that there is no particular reason to prefer one definition to another. To a remarkable extent, the theory of regular functions mimics that of holomorphic functions, including the representation by a Cauchy integral formula together with its classical consequences: [23] is a good, modern reference for the subject. A higher dimensional version of the theory was provided in [20], by means of a componentwise approach to regularity in many quaternionic variables, while many other interesting extensions have been considered in the literature, with a number of physical applications: a quite exhaustive picture may be obtained from the evergreen book [2] and the more recent [14] and [7].
In spite of that, while complex analysis is a common knowledge in the mathematical community, the quaternionic version is quite unpopular nowadays, and is even looked at suspiciously because of some strange algebraic features: the composition of regular maps may be not regular, and even the prototype of the good maps, i.e. the identity, is not a regular function. Many alternative definitions of regularity have been introduced in the literature, to overcome these problems: see for instance [4], [11], [12] and the more recent [9]. On the contrary, the scope of the present paper is within the classical Fueter theory, not with the aim of fixing the algebraic problems of the regular functions, but instead of understanding the algebraic reasons for them.
The main device is an algebraic description of the regularity itself. Along the process one discovers many interesting facts: for instance, that the choice of the Cauchy–Riemann equation is not at all a matter of taste, but it is driven by the *chirality context*. A vector space over $\mathbb{H}$ may be a left space or a right space: the choice between the two possibilities is named here the *chirality* of the space itself. The map $f$ above acts between two copies of $\mathbb{H}$, which of course may be seen as vector spaces over $\mathbb{H}$, each one with its own chirality: this gives rise to four different *chirality contexts* and the point is, that they are in one–to–one correspondence with the equations (1.1)–(1.2). A main point of this paper is to show that the chirality context is indeed the key to solve all the puzzles that the non commutativity of $\mathbb{H}$ involves. To this aim, while most of the literature begins with a concrete choice of the chirality context, this will not be done here: along all the paper, the name *quaternionic space* will denote a vector space over $\mathbb{H}$, without an priori choice for the chirality.
The chirality context drives the way the real differential of a map may be written as the sum of two parts which, roughly speaking, behave as the real and the imaginary part of a quaternion: the composition of maps is shown to have exactly the same algebraic structure as the product of quarternions. In this picture, the main result states that *regularity is the same as having a purely imaginary differential*, so that the bad properties of regularity simply mirror the unavoidable and widely accepted properties of the imaginary quaternions: the unit is not imaginary, nor is the product of imaginary quaternions, in general. Moreover, it will become clear that also the standard notion of holomorphy may have similar algebraic drawbacks, when settled in a quaternionic context: for linear approximations, see for instance the comments after Proposition 5.10.

After this preview, let me enter some more details abut the ideas and the results of the paper. In the complex case, the Cauchy–Riemann equations express the vanishing of one of the coefficients in the formula

$$df = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z} \tag{1.3}$$

and the basic question here is: is there anything similar for the equations (1.1)–(1.2), in the quaternionic case?
The question is not new, of course, and appears in the first pages of almost every introduction to quaternionic analysis. The typical way to attach the problem is to say that the two differential forms $dq$ and $d\overline{q}$ are not sufficient to span $\mathrm{Lin}^{\mathbb{R}}(\mathbb{H})$ over the quaternions and that, moreover, it seems difficult to complete them to a basis: see, for instance, [22] or [7]. Then one looks for other type of decompositions, where the Cauchy–Riemann equations nevertheless express the vanishing of some particular term. Some interesting answers are provided in [22] and [15]. The paper [22] is quite related to the present one, and it will be commented later on in the Introduction. In [15], the real differential of a regular function is shown to possess some additional linearity properties, with an intermediate homogeneity between the real and the quaternionic ones: roughly speaking, it is a complex linearity with respect to three independent, complex–type variables $x - it, y - jt, z - kt$. These variables are usually called Fueter variables, and have a natural extension in the context of Clifford algebras (see also [7]).
There is another obstruction to the word–by–word translation of (1.3) into the quaternionic framework. This formula is nothing else than the one–dimensional, differential guise of a general and well known purely algebraic fact: every real linear map, between complex spaces, can be uniquely decomposed into the sum of a complex linear map and a complex anti–linear one. The quaternionic analogue should then be concerned with quaternionic linear or anti–linear maps, which are not enough to decompose real linear maps but, even worse, are not suitable for a rich differential calculus: if $df$ is quaternionic linear at every point, then $f$ is affine. It is part of the mathematical folklore that this failure is the main reason for the Cauchy–Riemann approach to regularity. This fact has been proved so many times in the literature, in the one or in the finite dimensional cases, that I could not resist to add also my personal version of it: Proposition 6.4 provides a quite general statement, with a very short and elementary proof.
Summing up, there are a lot of evidences suggesting that quaternionic linearity is not relevant to quaternionic analysis: on the contrary, my hope is to convince the reader that, when correctly interpreted, it is the ultimate brick the regularity is made of. The preliminary step is to understand that quaternionic linear and anti–linear maps, as $dq$ and $d\overline{q}$, cannot stay together. This is better seen when working in $\mathrm{Lin}^{\mathbb{R}}(X;Y)$, where $X$ and $Y$ are general quaternionic spaces. While $\mathbb{H}$ may act on itself from the left and from the right, on general spaces the action is typically one–sided: denote it by $*$, whatever the chirality is. Quaternionic linear and anti–linear maps are particular instances of the following general notion: an additive map $\Lambda : X \to Y$ is said to be *quaternionic linear* with respect to a given $\varphi : \mathbb{H} \to \mathbb{H}$ if

$$\Lambda(q * x) = \varphi(q) * \Lambda(x)$$

for every $q \in \mathbb{H}$ and $x \in X$. The resulting class is denoted by $\mathrm{Lin}^{\mathbb{H}}_{\varphi}(X;Y)$, and its properties are studied in Section 3. It is worth to notice that the homogeneity with respect to $\mathbb{R}$ is not a priori required, but instead is an a posteriori feature of the quaternionic linearity. Indeed, while $\varphi$ can be any map when $\Lambda = 0$, as soon as $\Lambda \neq 0$ the Proposition 3.1 says that $\varphi$ must be a field automorphisms which preserves the order in the product, if $X$ and $Y$ have the same chirality, or which reverses it, in the opposite case. The conclusion is made by two parts. The first one is that $\varphi$ is not just a field morphism, but instead a field automorphism and hence also an automorphism of real algebras. This fact depends on Proposition 2.1 and says that $\varphi$ must fix $\mathbb{R}$, so explaining why

$$\mathrm{Lin}^{\mathbb{H}}_{\varphi}(X;Y) \subset \mathrm{Lin}^{\mathbb{R}}(X;Y) \; .$$

The second and more relevant part, though easier to prove, concerns the role played by the chirality context in the choice of the morphism type: reversing or not reversing the order in the quaternionic multiplication. This fact represents the deepest obstruction to the translation of the decomposition (1.3) into the quaternionic context: also when $X = Y = \mathbb{H}$ as sets, but the chirality context is given once for all, there are no chances to put $dq$ and $d\overline{q}$ into the same formula.
The class of the automorphisms of $\mathbb{H}$ will be denoted by $\mathrm{Aut}(\mathbb{H})$, and their properties are crucial for this paper. This class consists indeed of the inner automorphisms, defined by

$$\vartheta_g(q) = gqg^{-1}$$

for every given quaternion $g \neq 0$. The algebraic and differential properties of the map $\vartheta : \mathbb{H}\backslash\{0\} \to \mathrm{Aut}(\mathbb{H}) \subset \mathrm{Lin}^{\mathbb{R}}(\mathbb{H})$ are well described in almost every textbook about quaternions. For the purposes of this paper, however, the *metric* properties turn out to be relevant: they seem to be overlooked in the literature, and are studied in the final part of section Section 2. To express them, consider the standard real Euclidean scalar product on $\mathbb{H}$, and endow $\mathrm{Lin}^{\mathbb{R}}(\mathbb{H})$ with the unique *quaternionic* scalar product, having the standard real one (see (2.6)) as its real part. The crucial fact follows from Proposition 2.7 and is stated in Corollary 2.8: $\vartheta$ maps real orthogonal vectors of $\mathbb{H}$ into automorphisms which are quaternionic orthogonal in $\mathrm{Lin}^{\mathbb{R}}(\mathbb{H})$.
Roughly speaking, the effect of $\vartheta$ is to export into $\mathrm{Aut}(\mathbb{H})$, and then into $\mathrm{Lin}^{\mathbb{R}}(\mathbb{H})$ after a quaternionic span, the classical orthogonal decomposition $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}^3$ into the real and imaginary quaternions respectively: the real quaternions correspond to the old fashioned quaternionic linear or anti–linear maps, depending on the chirality context, while the second ones are exactly the regular maps. This is worked out in Section 4, for general quaternionic spaces. To state the result, consider a *standard basis* of $\mathbb{H}$, namely a real orthonormal basis $e_0, e_1, e_2, e_3$ with the following extra–properties: it satisfies $e_0 = 1$, i.e it respects the orthogonal decomposition of $\mathbb{H}$, and it is well oriented, in the sense that $e_1 e_2 = e_3$. The inner representation $\vartheta$ maps this basis into $\omega_0 = \mathrm{id}$ and three *imaginary* orthogonal automorphisms $\omega_1, \omega_2, \omega_3$. Their quaternionic orthogonality makes it easy to show that: every $\Lambda \in \mathrm{Lin}^{\mathbb{R}}(X : Y)$ may be uniquely decomposed into the sum

$$\Lambda \;=\; \Lambda_0 \;+\; \Big\{ \Lambda_1 + \Lambda_2 + \Lambda_3 \Big\} \tag{1.4}$$

where every $\Lambda_k$ is quaternionic linear with respect to $\varphi_k$ and

$$\varphi_k = \omega_k \qquad \text{or} \qquad \varphi_k = \overline{\omega}_k$$

according to whether $X$ and $Y$ have the same or different chiralities, respectively. Moreover, and of great relevance here, the coarser decomposition given by the parenthesis does not depend on the concrete choice of the imaginary units $e_1, e_2, e_3$. As anyone who has worked in the field knows, this quality is like a divide in the quaternionic world, separating the *canonical* objects from the non canonical ones: of course, the Cauchy–Riemann equations also belong to the first category.

The claim is that the decomposition (1.4) is the quaternionic analogue of the complex decomposition underlying (1.3), and that the condition

$$\Lambda_0 = 0 \tag{1.5}$$

defines a general, coordinate independent notion of regularity. The algebraic analogy of regular maps with the imaginary part of quaternions becomes manifest when looking at the canonical decomposition of a composition of maps: this is done at the end of section 4.

Notice that, depending on the chirality context, $\varphi_0$ is either the identity or the conjugation in $\mathbb{H}$. Thus condition (1.5) expresses the vanishing of either the quaternionic linear part of $\Lambda$ or its quaternionic anti–linear part. The same notion of regularity was already introduced in [8], by guessing the form of the Cauchy–Riemann equations in general spaces, without worrying about canonical procedures, and without providing any general notion of quaternionic linearity or quaternionic orthogonality. However, it is exactly the orthogonality argument which completes the parallel between the canonical decomposition and $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}^3$, extending it from the algebraic level to the metric one, at lest in the one–dimensional case. In this case indeed (1.4) becomes a true quaternionic orthogonal decomposition: since moreover each class $\mathrm{Lin}^{\mathbb{H}}_{\varphi}(\mathbb{H})$ is spanned by $\varphi$, over the quaternions, regularity reads exactly as the quaternionic orthogonality to $\varphi_0$.

Looking at the one–dimensional case, one also achieves the full justification of definition (1.5). By specializing the canonical decomposition to differential forms, the quaternionic version of (1.3) is found to be

$$df \; = \; \frac{\partial f}{\partial \varphi_0} \diamond d\varphi_0 \; + \; \sum_{h=1}^{3} \frac{\partial f}{\partial \varphi_h} \diamond d\varphi_h \, . \tag{1.6}$$

where the coefficients are the differential operators

$$\frac{\partial f}{\partial \varphi_h}(x) \; = \; \frac{1}{4} \sum_{k=0}^{3} \overline{\varphi}_h(e_k) * \left\{ df(x) \cdot e_k \right\} .$$

Here $*$ and $\diamond$ denote the two opposite ways $\mathbb{H}$ may act on itself: $*$ is decided by the initial chirality context and, as explained in Section 3, the use of $\diamond$ on $\mathrm{Lin}^{\mathbb{R}}(\mathbb{H})$

guarantees that the notion of quaternionic linearity is closed under quaternionic span. The regularity of $f$, as defined by (1.5), becomes the differential condition

$$\frac{\partial f}{\partial \varphi_0} = 0 .$$

To write it explicitly one must know the concrete expression of $*$ and $\varphi_0$ which, in turn, depend on the chiralities of the involved spaces. By choosing them, one obtains exactly the four equations (1.1)–(1.2): one for each chirality context. In other words, the notion of regularity is totally driven by the chirality context: though this fact should be quite easily guessed from the aspect of the Cauchy–Riemann equations, I was unable to trace it in the literature. The differential decomposition (1.6) should be compared with that of [22]. Without mentioning any general notion of quaternionic linearity, the same object $df$ is decomposed there along the same differential forms, but the coefficients turn out to be different: indeed, they are wrong in general, as explained in Remark 4.5.

Once (1.5) has been accepted as definition of regularity, the canonical decomposition provides a purely algebraic characterization of the regularity itself: a real linear map is regular if and only if is the sum of quaternionic linear maps, with respect to imaginary automorphisms. Three of them are sufficient for all the regular maps, but they are not always necessary: the minimal number of quaternionic linear maps, with respect to mutually orthogonal imaginary automorphisms, which are really needed to decompose a regular map $\Lambda$, deserves the name of *quaternionic size* of $\Lambda$, and is denoted by $s\{\Lambda\}$. This number turns out to have a very concrete algebraic meaning: while size one maps are (by definition) the regular quaternionic linear maps, and having size three does not operate any selection in the class of the regular maps, one has $s\{\Lambda\} \leq 2$ if and only if the regular map $\Lambda$ is *complex linear*, with respect to some suitable choice of the complex structures in $X$ and $Y$. Notice that there is not a canonical way to think of a quaternionic space as a complex one: there are as many complex structures, as the imaginary units are. In [19], another classification is obtained, by counting the number of different complex structures for which a given $\Lambda$ is complex linear: in fact, the two classifications are equivalent. These and other facts are proved in Section 5, yielding a full comprehension of the relationships between complex linearity and regularity.

Finally, Section 6 is devoted to the nonlinear consequences of the size restrictions. The main result is Theorem 6.1 and concerns the one–dimensional case. Roughly speaking, it says that truly nonlinear regular maps cannot be too simple from the algebraic point of view: indeed, the quaternionic size of its real differential must be strictly bigger than one in a everywhere dense subset. The key point here is the identification between quaternionic linearity and conformality, introduced by Lemma 6.2. On the one hand, this fact allows to use the classical Liouville's theorem on conformal maps: for the convenience of the reader, its proof is sketched in the Appendix B. On the other hand, this identification provides an alternative and more direct answer to a question raised in [13]: formula (6.6) shows how the conformality may be written in terms of difference quotients.

**Notation.** In this paper $\mathbb{H}$ is the quaternion field and Aut($\mathbb{H}$) is the class of the automorphisms of $\mathbb{H}$. The decomposition $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}^3$ is the canonical, real orthogonal decomposition of $\mathbb{H}$, into real and imaginary quaternions. The identity on $\mathbb{H}$ is denoted by id while cj stays for the classical conjugation. The symbols $p \cdot q$ and $p \times q$ denote respectively: the Euclidean real scalar product of general quaternions, and the vector product of imaginary quaternions.
Moreover $X, Y, Z$ stand for quaternionic spaces, namely vector (or Banach) spaces over $\mathbb{H}$, and $*$ is the associated scalar multiplication. Depending on the space, the scalar multiplication $*$ may define a left action of $\mathbb{H}$ or a right one: this quality is named the *chirality* of the corresponding space. The symbol $\diamond$ denotes a second quaternionic scalar multiplication (if any) which satisfies the compatiblity condition given by (3.4).
Also, $\mathrm{Lin}^{\mathbb{R}}(X;Y)$ is the class of the real linear maps from $X$ to $Y$, interpreted as real spaces, while the shorter notation $\mathrm{Lin}^{\mathbb{R}}(X)$ is used in case $Y = X$. Replacing $\mathbb{R}$ by $\mathbb{C}$ or $\mathbb{H}$ corresponds to changing the homogeneity rules.
All the (nonlinear) functions $f : X \to Y$ considered in the paper are just required to possess the real differential $df(x)$, at least for all $x$ in some open subset of $X$. The directional derivative of $f$ at $x$ along $v$ is denoted by $df(x) \cdot v$. The one–time real differentiability of $f$ is enough everywhere but some arguments in the final section: also there, however, the increased degree of smoothness is never an a priori assumtpion, but instead an a posteriori consequence of some suitable algebraic restrictions on $df$.
Finally, a warning is necessary about the *regularity* of a function: during all the paper, the word regularity does not allude to the degree of smoothness of the function, but instead to its quaternionic regularity.

## 2 Orthogonal automorphisms of $\mathbb{H}$

The quaternion field $\mathbb{H}$ is the unitary $\mathbb{R}$–algebra generated by the symbols $, i, j, k$ with the relations

$$\begin{gathered} i^2 = j^2 = k^2 = -1 \\ ij = -ji = k \qquad jk = -kj = i \qquad ki = -ik = j \,. \end{gathered}$$

In this way one obtains a 4–dimensional division algebra over the reals, whose multiplication is associative but not commutative. The generic element writes as

$$q = q_0 + q_1 i + q_2 j + q_3 k$$

where the coefficients are real, and most of the algebraic manipulations involve its conjugate

$$\overline{q} = q_0 - q_1 i - q_2 j - q_3 k \,.$$

The calculus rules may be found in any textbook and will not even be recalled here. It is only worth to mention that two natural projections may be defined

$$\mathrm{Re}\, q = (q + \overline{q}\,)/2 = q_0 \qquad \mathrm{Im}\, q = (q - \overline{q}\,)/2 = q_1 i + q_2 j + q_3 k \,,$$

into the real and the imaginary part of $q$. They are orthogonal with respect to the standard scalar product of $\mathbb{R}^4$

$$q \cdot p = \mathrm{Re}\,(q\overline{p}\,)$$

and then, after the identifications Re $\mathbb{H} \cong \mathbb{R}$ and Im $\mathbb{H} \cong \mathbb{R}^3$, they induce the orthogonal decomposition

$$\mathbb{H} = \mathbb{R} \oplus \mathbb{R}^3 \,. \tag{2.1}$$

The first identification is a real algebra isomorphism. Also notice that $\mathbb{R}$ is exactly the center of $\mathbb{H}$ itself. The second identification is just an isomorphism of real vector spaces, which however allows to rewrite the multiplication of quaternions in terms of the standard vector operations in $\mathbb{R}^3$: being clear how to multiply a real number with a vector, everything is determined by the rule

$$u\,v \;=\; -u \cdot v + u \times v \tag{2.2}$$

for every $u, v \in \mathbb{R}^3$. Notice that, in particular, if $p = t + u$ with $t \in \mathbb{R}$ and $u \in \mathbb{R}^3$, then one has

$$p^2 = t^2 - u \cdot u + 2tu \,.$$

This number is real if and only if either $t = 0$ or $u = 0$, a fact which yields a purely algebraic characterization of the decomposition (2.1)

$$p \in \mathbb{R} \iff p^2 \geq 0 \qquad\qquad p \in \mathbb{R}^3 \iff p^2 \leq 0 \,.$$

In other words, the decomposition (2.1), with the associated notions of real and imaginary quaternion, is uniquely determined by the algebraic property of $\mathbb{H}$. What is not canonical here, is just the choice of the imaginary units: $i$, $j$ and $k$ do not play any privileged role, every equally oriented orthonormal basis of $\mathbb{R}^3$ doing exactly the same job. By adding $1 \in \mathbb{R}$ to them, one obtains a special class of real bases for $\mathbb{H}$: trough this paper, they will be referred as the *standard bases* of $\mathbb{H}$. Sometimes, a standard basis is chosen to have a suitable coordinates description, but basic notions in the theory of quaternions must be independent of it. For instance, it is not difficult to check that this is the case of condition (1.1). Notions which respect this invariance will be said to be *canonical*.
The main canonical object of this paper will be introduced in Section 4: its construction requires a good understanding of the automorphisms of $\mathbb{H}$, which are the true topic of this section. A field morphism is a map $\omega : \mathbb{H} \to \mathbb{H}$ which satisfies

$$\omega(p+q) = \omega(p) + \omega(q) \qquad \omega(p\,q) = \omega(p)\,\omega(q) \qquad \omega(1) = 1$$

for all $p, q \in \mathbb{H}$. Because of the first two requests, the kernel of $\omega$ must be a two–sided ideal of $\mathbb{H}$: the last requirement then expresses the nontriviality of $\omega$ or, equivalently, its injectivity. In particular, $\omega$ must preserve the multiplicative inverses, besides the additive ones, which in turn yields

$$\omega(t) = t \qquad\quad \forall t \in \mathbb{Q} \,.$$

Whether this is true for every real $t$ or not, depends on the continuity of $\omega$: if this is the case, then $\omega$ is said to be a *real algebra morphism*.
It is worth to notice that, when field morphisms of $\mathbb{R}$ are considered, continuity comes for free (so that they reduce to the identity alone), while many field morphisms of $\mathbb{C}$ exist which are discontinuous: an elementary and complete discussion about these well known bits of mathematical folklore may be found in [24]. For this paper it is a relevant fact that the quaternionic case behaves exactly as the real one: a proof is provided, just because I was unable to trace it in the literature.

**Proposition 2.1.** *Every field morphism of $\mathbb{H}$ is the identity on $\mathbb{R}$, and then a continuous real algebra automorphism.*

*Proof.* Denote by $\omega$ any given field morphism of $\mathbb{H}$. The first conclusion says that $\omega$ is a real linear map: since it is already known to be injective, the remaining conclusions follow from standard arguments in the theory of linear maps.
To prove the first conclusion, one has to show that $\omega(\mathbb{R}) \subset \mathbb{R}$. This would be clear if $\omega$ were surjective, since $\mathbb{R}$ is the center of $\mathbb{H}$. In fact, it is also true if the range of $\omega$ is big enough. To see why, begin by noticing that

$$\omega(t)\,\omega(q) = \omega(t\,q) = \omega(q\,t) = \omega(q)\,\omega(t)$$

for every $t \in \mathbb{R}$ and $q \in \mathbb{H}$. Now, (2.2) says that the two quaternions commute if and only if the cross product of their imaginary parts vanishes. Hence, $\omega(t)$ must be real in the previous formula as soon as Im $\omega(\mathbb{H})$ contains at least two real independent vectors. This follows again from (2.2), by applying $\omega$. Indeed, by taking first $p = q \in \mathbb{R}^3$ one has

$$\omega(q)^2 = \omega(q^2) = \omega(-1) = -1$$

showing that the unitary sphere of $\mathbb{R}^3$ is mapped into itself. Then, by taking unitary and orthogonal $p, q \in \mathbb{R}^3$, one has

$$\omega(p)\,\omega(q) + \omega(q)\,\omega(p) = \omega(p\,q + q\,p) = \omega(0) = 0$$

showing that $\omega$ maps them into orthogonal vectors. ∎

The automorphisms of $\mathbb{H}$ will be denoted by Aut($\mathbb{H}$): the previous proposition says that there is no need to specify whether they preserve the field or the real algebra structure. Among the automorphisms, the *inner* ones are defined by

$$\vartheta_g(q) = g\,q\,g^{-1} \qquad\qquad q \in \mathbb{H}$$

where $g$ is some nonzero quaternion. The map $g \mapsto \vartheta_g$ is a multiplicative morphism (with respect to the composition of maps) having kernel $\mathbb{R}\setminus\{0\}$. The explicit definition makes evident some extra–features of the inner automorphisms, which are described in the next lemma: a proof is provided just for the reader convenience.

**Lemma 2.2.** *For every nonzero $g$, the automorphism $\vartheta_g$ leaves $\mathbb{R}^3$ invariant and is a positive rotation around* $\operatorname{Im} g$ *of an angle $\alpha$ determined by*

$$\cos(\alpha) = \frac{|\operatorname{Re} g|^2 - |\operatorname{Im} g|^2}{|\operatorname{Re} g|^2 + |\operatorname{Im} g|^2} \qquad \sin(\alpha) = \frac{2 \operatorname{Re} g \, |\operatorname{Im} g|}{|\operatorname{Re} g|^2 + |\operatorname{Im} g|^2} \, .$$

*Proof.* Since $x^2 \leq 0$ yields $\vartheta_g(x)^2 = gx^2g^{-1} = x^2 \leq 0$, the first conclusion follows. Moreover, $\vartheta_g(\operatorname{Im} g) = \operatorname{Im} g$ while, when $v \in \mathbb{R}^3$ is orthogonal to $\operatorname{Im} g$, one has

$$\vartheta_g(v) = \frac{g^2}{|g|^2} v = \frac{|\operatorname{Re} g|^2 - |\operatorname{Im} g|^2}{|\operatorname{Re} g|^2 + |\operatorname{Im} g|^2} v + \frac{2 \operatorname{Re} g}{|\operatorname{Re} g|^2 + |\operatorname{Im} g|^2} \operatorname{Im} g \times v \, . \qquad \blacksquare$$

Of course, this way one may describe all the positive rotations of $\mathbb{R}^3$. This leads to the following important and well known fact.

**Proposition 2.3.** *All the automorphisms of* $\mathbb{H}$ *are inner automorphisms.*

*Proof.* It remains to show that every $\omega \in \operatorname{Aut}(\mathbb{H})$ is indeed a positive rotation of $\mathbb{R}^3$. This may be done by arguing from (2.2), exactly as in the final part of the proof of Proposition 2.1: the difference is that now one knows that all the reals are left unchanged, not only 0 and 1. The positivity of the rotation follows from the conservation of the cross product. $\blacksquare$

The inner representation of an automorphism is very convenient for the aims of this paper. Quite clearly, it makes computations more concrete. For instance, the problem to find which automorphisms are involutive reduces to determine which $g$ satisfy

$$g^2 q g^{-2} = q \qquad \forall q \in \mathbb{H} \, .$$

This is equivalent to ask that $g^2 \in \mathbb{R}$, namely that: either $g \in \mathbb{R}$ or $g \in \mathbb{R}^3$. In the first case, the automorphism is the identity. In the second one is a rotation around $g$ in $\mathbb{R}^3$ of an angle $\pi$: it will be referred as an *imaginary automorphism*.
Another consequence of the inner representation is to provide a convenient notation for the anti–automorphisms of $\mathbb{H}$: they reverse the order in a product, instead of maintaining it. The conjugation is the most obvious example of anti–automorphisms, and all of them are clearly obtained by composing it with the automorphisms. Though in principle this composition may be done in two different ways, an explicit computation yields

$$\overline{\vartheta_g(q)} = \overline{g^{-1}}\,\overline{q}\,\overline{g} = g\,\overline{q}g^{-1} = \vartheta_g(\overline{q}) \tag{2.3}$$

for every $g$ and $q$, showing the non ambiguity of the notation

$$\overline{\omega} \qquad \text{where} \qquad \omega \in \operatorname{Aut}(\mathbb{H})$$

to denote the generic anti–automorphism of $\mathbb{H}$. It is not difficult to check that $\overline{\omega}$ is involutive if and only if $\omega$ is: by analogy, also $\overline{\omega}$ will be called imaginary when $\omega$ is.
A further advantage of the inner representation is to obtain an explicit description of the automorphisms $\omega$ which satisfy

$$\omega(p) = q$$

for given $p, q \in \mathbb{H}$: this will turn out to be useful in Section 5. In order for the equation to be solvable, $p$ and $q$ must have the same modulus and, as $\omega$ is the identity on $\mathbb{R}$, they also must have the same real part: then it's enough to consider imaginary units. Thinking of $\omega = \vartheta_g$, the answer is provided by the following lemma.

**Lemma 2.4.** *Let $p$ and $q$ be imaginary units, and consider the equation $gp = qg$. When $p + q = 0$, the solutions are*

$$g \in \mathbb{R}^3 \qquad \textit{such that} \qquad g \cdot p = 0 \tag{2.4}$$

*whereas, when $p + q \neq 0$, they are*

$$g = \lambda(p+q) + \mu \left\{ 1 + \frac{2}{|p+q|^2}\, p \times q \right\} \qquad \textit{where} \qquad \lambda, \mu \in \mathbb{R}. \tag{2.5}$$

*Proof.* After writing $g = \mu + u$, with $\mu \in \mathbb{R}$ and $u \in \mathbb{R}^3$, the equation $gp = qg$ splits into the two conditions

$$(p - q) \cdot u = 0 \qquad\qquad (p+q) \times u = \mu(p-q)\ .$$

Notice that $(p + q) \cdot (p - q) = 0$, so that the second equation is always solvable. When $p + q = 0$, the second equation yields $\mu = 0$ and the thesis is just the transcription of the first equation. On the other hand, when $p + q \neq 0$ the first equation follows from the second one. The kernel of the second equations is given by the real multiples of $p + q$, while the general solution orthogonal to the kernel may be easily computed via the quaternionic product. Indeed one finds

$$u = \mu(p+q)^{-1}(p-q) = \frac{\mu}{|p+q|^2}\,(p-q) \times (p+q) = \frac{2\mu}{|p+q|^2}\, p \times q$$

which concludes the proof. ∎

Beside the computational convenience, the inner representation allows also to transfer the notion of standard basis from $\mathbb{H}$ to the space $\mathrm{Lin}^{\mathbb{R}}(\mathbb{H})$ of the real linear mappings from $\mathbb{H}$ into itself . Why this is relevant for regularity, it will be explained in section 4: the aim hereafter is simply to justify the new notion, by making use of the underlying orthogonal structure.
The standard *real* scalar product on $\mathrm{Lin}^{\mathbb{R}}(\mathbb{H})$ is given by

$$\langle \Lambda \,|\, \Gamma \rangle \;=\; \frac{1}{4} \sum_{k=0}^{3} \Lambda(e_k) \cdot \Gamma(e_k) \;=\; \frac{1}{4}\, \mathrm{trace} \left( \mathbf{\Lambda}^T\, \mathbf{\Gamma} \right) \tag{2.6}$$

where $e_0, e_1, e_2, e_3$ is any positively oriented, orthonormal basis of $\mathbb{H}$ as a real space. The last form, where the boldface is used to denote the associated matrices, is probably best known. It is not difficult to check that the inner automorphisms associated to a standard basis $e_0, e_1, e_2, e_3$ are indeed orthonormal with respect to the scalar product (2.6). From one hand this certainly suggests that they are good candidates to be a standard basis of $\mathrm{Lin}^{\mathbb{R}}(\mathbb{H})$. On the other hand, it is clear that these automorphisms cannot span $\mathrm{Lin}^{\mathbb{R}}(\mathbb{H})$ over real coefficients, by a dimensional argument. Quaternionic coefficients are needed for that, which in turn asks for a stronger notion of orthogonality, namely with respect to a *quaternionic* scalar product. These scalar products are defined as in the complex case, but for some automatic restrictions imposed by the non commutativity of $\mathbb{H}$: see the Appendix B for more details. The one–dimensional prototype is

$$p\,\overline{q} \qquad \text{or} \qquad \overline{p}\,q$$

according to the chirality context: the first one has the correct homogeneity when $\mathbb{H}$ is interpreted as a left space over itself, while the second one works for right spaces. Of course, by conjugating one of them, another scalar product is obtained for the same chirality. In any case, all these scalar products extend the standard real scalar product in $\mathbb{H}$: next lemma does the same for the space of real linear maps.

**Lemma 2.5.** *There exists a unique (up to conjugation) quaternionic scalar product in* $\mathrm{Lin}^{\mathbb{R}}(\mathbb{H})$, *having (2.6) as its real part, namely*

$$\left(\Lambda \,|\, \Gamma\right)_l = \frac{1}{4}\sum_{k=0}^{3} \Lambda(e_k)\,\overline{\Gamma(e_k)} \qquad \textit{or} \qquad \left(\Lambda \,|\, \Gamma\right)_r = \frac{1}{4}\sum_{k=0}^{3} \overline{\Lambda(e_k)}\,\Gamma(e_k) \tag{2.7}$$

*according to whether* $\mathrm{Lin}^{\mathbb{R}}(\mathbb{H})$ *is considered as a left or a right quaternionic space, respectively.*

Here $e_0, e_1, e_2, e_3$ denotes again an arbitrary real orthonormal basis of $\mathbb{H}$: as for (2.6), also (2.7) is independent of the concrete choice of the basis. Moreover, notice that conjugation acts on the scalar product as a change in the chirality of the space, in the sense that

$$\left(\overline{\Lambda} \,|\, \overline{\Gamma}\right)_l = \left(\Lambda \,|\, \Gamma\right)_r \tag{2.8}$$

for every $\Lambda, \Gamma \in \mathrm{Lin}^{\mathbb{R}}(\mathbb{H})$.

*Proof.* The uniqueness is the only nontrivial claim: it will be proved for the left case only, since the right one is analogous. Assume to deal with standard bases. The trivial identity

$$q = \sum_{h=0}^{3} \overline{e_h}\,\mathrm{Re}\,(e_h q) \tag{2.9}$$

allows to express the quaternionic scalar product in terms of the real part only

$$\left(\Lambda \,|\, \Gamma\right)_l = \sum_{h=0}^{3} \overline{e_h}\,\mathrm{Re}\left\{\left(e_h\Lambda \,|\, \Gamma\right)_l\right\} = \sum_{h=0}^{3} \overline{e_h}\,\left\langle e_h\Lambda \,|\, \Gamma\right\rangle .$$

Here $\left(q\Lambda \,|\, \Gamma\right)_l$ has been computed as $q\left(\Lambda \,|\, \Gamma\right)_l$, instead of $\left(\Lambda \,|\, \Gamma\right)_l\overline{q}$: the last choice leads to the conjugate result. For right structures, the normalization condition leading to (2.7) is $\left(\Lambda \,|\, \Gamma q\right)_r = \left(\Lambda \,|\, \Gamma\right)_r q$.

To conclude the proof, it is now sufficient to use the concrete expression (2.6) of the real scalar product. Indeed, one gets

$$\begin{aligned}(\Lambda \mid \Gamma)_l &= \sum_{h=0}^{3} \overline{e_h}\, \frac{1}{4} \sum_{k=0}^{3} \mathrm{Re}\, \{e_h \Lambda(e_k)\, \overline{\Gamma(e_k)}\} = \frac{1}{4} \sum_{k=0}^{3} \sum_{h=0}^{3} \overline{e_h}\, \mathrm{Re}\, \{e_h \Lambda(e_k)\, \overline{\Gamma(e_k)}\} \\ &= \frac{1}{4} \sum_{k=0}^{3} \Lambda(e_k)\, \overline{\Gamma(e_k)}\end{aligned}$$

where the last equality depends again on (2.9). ■

*Remark* 2.6. The above lemma has of course a complex analogous. When maps in $\mathrm{Lin}^{\mathbb{R}}(\mathbb{C})$ are considered, the analogous of the real scalar product (2.6) is a two term average with $e_0 = 1$ and $e_1 = i$. Up to conjugation, this is the real part of the complex scalar product

$$(\Lambda \mid \Gamma) \;=\; \frac{1}{2}\Big\{\Lambda(1)\overline{\Gamma(1)} + \Lambda(i)\overline{\Gamma(i)}\Big\}\,.$$

This will be considered in section 4, for comparative reasons only.

Next lemma describes how the quaternionic scalar product in $\mathrm{Lin}^{\mathbb{R}}(\mathbb{H})$ behaves, with respect to the inner representation of the automorphisms of $\mathbb{H}$. This result is a key fact for the purposes of this paper.

**Proposition 2.7.** *The formula*

$$(\vartheta_g \mid \vartheta_h)_l \;=\; \Big\{\mathrm{Re}\,(g^{-1}h)\Big\}\; g\,h^{-1} \;=\; (\vartheta_g \mid \vartheta_h)_r$$

*holds for every pair of nonzero quaternions $g$ and $h$.*

Since it may be easily checked that

$$\mathrm{Re}\,(g^{-1}h) \;=\; \frac{g\cdot h}{|g|^2}\,,$$

one may deduce the following important consequence.

**Corollary 2.8.** *The automorphisms $\vartheta_g$ and $\vartheta_h$ are orthogonal in the quaternionic sense if and only if the quaternions $g$ and $h$ are in the real sense.*

Moreover, exactly the same is true for the anti–automorphisms $\overline{\vartheta_g}$ and $\overline{\vartheta_h}$. Indeed, because of (2.8) one has

$$(\overline{\omega} \mid \overline{\eta})_l = (\omega \mid \eta)_r = (\omega \mid \eta)_l = (\overline{\omega} \mid \overline{\eta})_r \tag{2.10}$$

for every pair of automorphisms $\omega$ and $\eta$. The common value of all the above scalar products will be denoted by $(\omega \mid \eta)$, without any reference to chiralities. Finally, notice that, because of the choice of the normalization factors in (2.7), one has

$$(\omega \mid \omega) = 1$$

for every automorphism $\omega$.

*Proof.* The commutation property (2.3) yields

$$\left(\vartheta_g \mid \vartheta_h\right)_l = \frac{1}{4}\sum_{k=0}^{3} \vartheta_g(e_k)\,\vartheta_h(\overline{e_k}) = \frac{1}{4}\sum_{k=0}^{3} \vartheta_g(\overline{e_k})\,\vartheta_h(e_k) = \left(\vartheta_g \mid \vartheta_h\right)_r$$

where a standard basis has been used, so that $\overline{e_0} = e_0 = 1$ and $\overline{e_k} = -e_k$ for the remaining $k$'s. Continuing the computation, one finds

$$\left(\vartheta_g \mid \vartheta_h\right)_l = \frac{1}{4} g \left\{\sum_{k=0}^{3} e_k\,(g^{-1}h)\,\overline{e_k}\right\} h^{-1} = \frac{1}{4} g \left\{\sum_{k=0}^{3} \vartheta_{e_k}(g^{-1}h)\right\} h^{-1}\,,$$

so that to conclude the proof it is enough to show that

$$\frac{1}{4}\sum_{k=0}^{3} \vartheta_{e_k}(q) = \operatorname{Re} q$$

for every $q \in \mathbb{H}$. To this aim, write $q = t + v$, with $t \in \mathbb{R}$ and $v \in \mathbb{R}^3$, and compute as follows

$$\sum_{k=0}^{3} \vartheta_{e_k}(t+v) = \vartheta_{e_0}(t+v) + \sum_{k=1}^{3} \vartheta_{e_k}(t) + \sum_{k=1}^{3} \vartheta_{e_k}(v) = t + v + 3t - v$$

where the last term is obtained by decomposing $v$ along $e_1, e_2, e_3$ and using that

$$\vartheta_{e_k}(e_h) \;=\; \begin{cases} e_k & \text{if } h = k \\ -e_h & \text{if } h \neq k \end{cases}$$

for every $h, k \neq 0$. ∎

Summing up, the inner representation

$$\vartheta : \mathbb{H} \setminus 0 \to \operatorname{Aut}(\mathbb{H}) \subset \operatorname{Lin}^{\mathbb{R}}(\mathbb{H})$$

maps *real orthogonal* quaternions into *quaternionic orthogonal* automorphisms. For instance, the imaginary automorphisms may be now described as the $\omega \in \operatorname{Aut}(\mathbb{H})$ which satisfy

$$\left(\omega \mid \mathrm{id}\right) = 0\,.$$

By a dimensional argument, real orthogonal bases of $\mathbb{H}$ are mapped into quaternionic orthogonal bases of $\operatorname{Lin}^{\mathbb{R}}(\mathbb{H})$: they will represent a quite convenient way to approach quaternionic linear theory, which lacks some of the very convenient features of the commutative case (see, for instance, [26]).
Starting from a standard basis of $\mathbb{H}$, one obtains: the identity id and three orthogonal automorphisms $\omega_1, \omega_2, \omega_3$. As $\vartheta$ is a group morphism, the composition rules of these automorphisms are obtained from the product rules of the associated standard bases of $\mathbb{H}$. They are

$$\begin{aligned} &\omega_1 \circ \omega_1 = \omega_2 \circ \omega_2 = \omega_3 \circ \omega_3 = \mathrm{id} \\ &\omega_1 \circ \omega_2 = \omega_3 \quad \omega_2 \circ \omega_3 = \omega_1 \quad \omega_1 \circ \omega_3 = \omega_2 \end{aligned} \tag{2.11}$$

where the orientation problems disappear because of the kernel of $\vartheta$. By analogy with $\mathbb{H}$, these quaternionic bases of automorphisms deserve the name of *standard bases* of $\mathrm{Lin}^{\mathbb{R}}(\mathbb{H})$. Sometimes, constructing an object may require to fix one of these bases: once again, canonical objects are those which do not change when the standard basis is varied. The most relevant example, for the purposes of this paper, will be considered in Section 4.

## 3 Quaternionic linearity

Let $X, Y$ be linear spaces over $\mathbb{H}$, and denote by the same symbol $*$ the corresponding scalar multiplications. An additive map $\Lambda : X \to Y$ is usually said to be quaternionic linear when

$$\Lambda(q * x) = q * \Lambda(x) \tag{3.1}$$

for every $x \in X$ and $q \in \mathbb{H}$. Two facts, however, suggest to relax the standard homogeneity condition (3.1). The first one is that, when $X$ and $Y$ have opposite chiralities, the resulting notion of linearity is totally useless: next proposition says that the only quaternionic linear map would be the trivial one. The second fact is more relevant will be clear after the next section: when $X$ and $Y$ have the same chirality, quaternionic linear maps are as abundant as in the commutative framework, but they cannot even contribute to regular maps.

The required relaxed version of quaternionic linearity is obtained by replacing the homogeneity condition (3.1) with

$$\Lambda(q * x) = \varphi(q) * \Lambda(x) \tag{3.2}$$

where $\varphi : \mathbb{H} \to \mathbb{H}$ is now any given map. The resulting notion of quaternionic linearity will be called *$\varphi$–linearity*, or linearity with respect to $\varphi$, and the associated class of maps will be denoted by $\mathrm{Lin}^{\mathbb{H}}_{\varphi}(X;Y)$: the map $\varphi$ itself will be referred as the *reference map* of this class.

Next proposition highlights a key property of this relaxed notion: though there are no *a priori* restrictions, the definition makes sense only for a very restricted class of $\varphi$'s.

**Proposition 3.1.** *Assume that* $\mathrm{Lin}^{\mathbb{H}}_{\varphi}(X;Y) \neq \{0\}$. *Then either $\varphi$ is an automorphism of $\mathbb{H}$ or $\overline{\varphi}$ is, depending on whether $X$ and $Y$ have or have not the same chirality.*

It may happen that the spaces $X$ and $Y$ admit a double quaternionic structure, a left one and a right one, as it happens for $\mathbb{H}$ itself. Before talking about quaternionic linearity, one has to specify a concrete choice for the involved chiralities: this choice will be referred as to the *chirality context* of the problem. In this terms, the lemma says that the chirality context decides the morphism type of $\varphi$: in the rest of the paper, when considering the space $\mathrm{Lin}^{\mathbb{H}}_{\varphi}(X;Y)$ this compatibility condition will be implicitly assumed to hold.

*Proof.* Let $\Lambda$ be $\varphi$–linear and $x \in X$ such that $\Lambda(x) \neq 0$. The additivity of $\Lambda$ yields

$$\varphi(p+q) * \Lambda(x) = (\varphi(p) + \varphi(q)) * \Lambda(x)$$

for every $p, q \in \mathbb{H}$ and then the additivity of $\varphi$. That $\varphi(1) = 1$ may be proved in a similar way. Concerning the product, assume for definiteness that $X$ and $Y$ are a right and a left space, respectively: all the other cases are similar. There results

$$\begin{aligned}\varphi(p\,q) * \Lambda(x) &= \Lambda\big(((p\,q) * x)\big) = \Lambda\big(q * (p * x)\big) \\ &= \varphi(q) * \big(\varphi(p) * \Lambda(x)\big) = \big(\varphi(q)\,\varphi(p)\big) * \Lambda(x)\end{aligned}$$

for every $p, q \in \mathbb{H}$. Summing up, $\varphi$ must be a field morphism of $\mathbb{H}$, but for reversing the order in the product, namely: $\overline{\varphi}$ is a field morphism. Proposition 2.1 then allows to conclude. ∎

Since every automorphism of $\mathbb{H}$ fixes $\mathbb{R}$, from the proposition one immediately deduces that

$$\mathrm{Lin}_{\varphi}^{\mathbb{H}}(X;Y) \subset \mathrm{Lin}^{\mathbb{R}}(X;Y)$$

whatever $\varphi$ is. In the next section it will be proved that the larger space may be obtained as a sum of the smaller ones, for different $\varphi$'s: this fact is relevant to the notion of regularity, which will be also investigated in the next section. The aim, hereafter, is just to point out some of the algebraic features of the quaternionic linearity. The first remark concerns the uniqueness of the reference morphism.

**Lemma 3.2.** *Assume that $\Lambda \in \mathrm{Lin}^{\mathbb{R}}(X;Y)$ is quaternionic linear with respect to $\varphi$ and with respect to $\psi$. Then either $\Lambda = 0$ or $\varphi = \psi$.*

*Proof.* Indeed $\big(\varphi(q) - \psi(q)\big) * \Lambda(x) = \Lambda(q * x) - \Lambda(q * x) = 0$ for every $q \in \mathbb{H}$ and $x \in X$. ∎

Another trivial property is that the reference morphism behaves naturally with respect to composition, namely: if $\Lambda \in \mathrm{Lin}_{\varphi}^{\mathbb{H}}(X;Y)$ and $\Gamma \in \mathrm{Lin}_{\psi}^{\mathbb{H}}(Y;Z)$ then

$$\Gamma \circ \Lambda \in \mathrm{Lin}_{\psi\circ\varphi}^{\mathbb{H}}(X;Z)\ . \tag{3.3}$$

In particular, if $\Lambda \in \mathrm{Lin}_{\varphi}^{\mathbb{H}}(X;Y)$ is invertible, then $\Lambda^{-1}$ is also quaternionic linear with respect to $\varphi^{-1}$. To this aim, notice that the morphisms type of $\varphi^{-1}$ is the same of $\varphi$, namely: the compatibility with the chirality context is preserved under the inversion. When $X = Y$, the class $\mathrm{Lin}_{\varphi}^{\mathbb{H}}(X)$ is closed under inversion if and only if $\varphi$ is involutive.

Consider now $\mathrm{Lin}_{\varphi}^{\mathbb{H}}(X;Y)$ as a vector subspace of $\mathrm{Lin}^{\mathbb{R}}(X;Y)$. It is not difficult to check that the latter space is a quaternionic space with respect to the pointwise operations inherited from $Y$: this way, it has the same chirality of $Y$. The question is to decide whether $\mathrm{Lin}_{\varphi}^{\mathbb{H}}(X;Y)$ is or is not a quaternionic subspace of $\mathrm{Lin}^{\mathbb{R}}(X;Y)$. Of course, no problems arise with sums or combinations with real coefficients, but things become worse when quaternionic coefficients are involved. Assume indeed that $p \in \mathbb{H}$ and $\Lambda \in \mathrm{Lin}_{\varphi}^{\mathbb{H}}(X;Y)$. Then $p * \Lambda$ is again quaternionic linear with respect to $\varphi$ if and only if

$$p * \big(\varphi(q) * \Lambda(x)\big) = \varphi(q) * \big(p * \Lambda(x)\big)$$

for every $q \in \mathbb{H}$ and $x \in X$. When $\Lambda$ is nontrivial, this implies

$$p\ \varphi(q) = \varphi(q) p$$

for every $q \in \mathbb{H}$ and, since $\varphi$ is bijective, this is possible only for real $p$'s. In other words, it seems there are no chances for $\mathrm{Lin}_{\varphi}^{\mathbb{H}}(X;Y)$ to be a quaternionic subspace of $\mathrm{Lin}^{\mathbb{R}}(X;Y)$.

In fact, there is a way to overcome the problem when a second quaternionic scalar multiplication, call it $\diamond$, is defined on $Y$ and fulfills the compatibility condition

$$p \diamond (q * y) = q * (p \diamond y) \tag{3.4}$$

for every $p, q \in \mathbb{H}$ and every $y \in Y$. The idea is to use again $*$ to define the notion of quaternionic linearity, but to think of $\diamond$ as defining the quaternionic structure of $\mathrm{Lin}^{\mathbb{R}}(X;Y)$. The result is that now $p \diamond \Lambda \in \mathrm{Lin}_{\varphi}^{\mathbb{H}}(X;Y)$ for every $p \in \mathbb{H}$, and is obtained by simply specializing (3.4).

*Remark* 3.3. Since the chirality of a quaternionic space $Y$ may be changed in a standard way, by defining $q \diamond y = \overline{q} * y$, the reader might have the doubt that all the above program may be worked out in the general case. However, it is not difficult to see that condition (3.4) never holds for this choice of $\diamond$.

The model case for the compatibility condition is $Y = \mathbb{H}$, with $*$ and $\diamond$ the two natural quaternionic scalar multiplications, namely

$$q * y = q\,y \qquad\qquad p \diamond y = yp$$

or viceversa. Then condition (3.4) writes down as

$$p\,(yq) = (py)q$$

for every $p$, $q$ and $y$ in $\mathbb{H}$: this is no longer a commutation property, but instead expresses the associativity of the quaternionic multiplication. This yields the following conclusion.

**Lemma 3.4.** *The space* $\mathrm{Lin}_{\varphi}^{\mathbb{H}}(X;\mathbb{H})$ *is a quaternionic subspace of* $\mathrm{Lin}^{\mathbb{R}}(X;\mathbb{H})$*, provided the two classes refer to different chiralities of* $\mathbb{H}$*.*

When also $X = \mathbb{H}$, some trivial, but relevant, consequences my be deduced from Lemma 3.4. The first one concerns the consistency of the classes of the quaternionic linear maps: as in the commutative case, it results

$$\mathrm{Lin}_{\varphi}^{\mathbb{H}}(\mathbb{H}) = \mathbb{H} \diamond \varphi \tag{3.5}$$

due to the standard argument

$$\Lambda(x) = \Lambda(x * 1) = \varphi(x) * \Lambda(1) = \Lambda(1) \diamond \varphi(x) = [\Lambda(1) \diamond \varphi](x)\ .$$

The second one concerns the propagation of the orthogonality, from the reference morphisms to the associated classes of quaternionic linear maps.

**Lemma 3.5.** *Let the morphisms $\varphi$ and $\psi$ refer to the same chirality context. Then*

$$(\varphi \mid \psi) = 0 \qquad \textit{implies} \qquad (\Lambda \mid \Gamma) = 0$$

*for every* $\Lambda \in \mathrm{Lin}_{\varphi}^{\mathbb{H}}(\mathbb{H})$ *and* $\Gamma \in \mathrm{Lin}_{\psi}^{\mathbb{H}}(\mathbb{H})$*.*

The lack of references to chiralities in the quaternionic scalar products is intentional. Concerning the reference morphisms, formula (2.10) guarantees that the notation is unambiguous. This is no longer true for $\big(\Lambda \mid \Gamma\big)$, in which case the ambiguity is solved by the chirality context: the scalar product must have the same chirality of the operation $\diamond$ in the codomain, namely the opposite chirality of the operation $*$ which has been used to define the notion of quaternionic linearity. The same convention will be used in the rest of the paper.

*Proof.* This is a trivial consequence of the quaternionic homogeneity of the scalar product, as soon as the aforementioned rule on the chiralities is respected. To make it explicit at least once, choose for instance to work with two left spaces. Then quaternionic linearity makes sense with respect to the automorphisms of $\mathbb{H}$. Moreover, since the codomain is a left space, $\mathrm{Lin}^{\mathbb{R}}(\mathbb{H})$ has to be considered as a right quaternionic space. Hence

$$\mathrm{Lin}^{\mathbb{H}}_{\omega}(\mathbb{H}) = \omega\, \mathbb{H}$$

for every $\omega \in \mathrm{Aut}(\mathbb{H})$. If now $\eta$ is another automorphism, then a direct computation shows that

$$\big(\omega\, p \mid \eta\, q\big)_r \;=\; \overline{p}\,(\omega \mid \eta)\, q$$

for every $p, q \in \mathbb{H}$. ∎

At a purely formal level, it is convenient to extend the notion of orthogonality from the one–dimensional case to the general one: given the quaternionic spaces $X$ and $Y$, and two compatible morphisms $\varphi$ and $\psi$, the maps $\Lambda \in \mathrm{Lin}^{\mathbb{H}}_{\varphi}(X;Y)$ and $\Gamma \in \mathrm{Lin}^{\mathbb{H}}_{\psi}(X;Y)$ will be called *orthogonal* when $\varphi$ and $\psi$ are. In the same spirit, a quaternionic linear map will be said *imaginary* when its reference morphism is.

## 4 Canonical decomposition of real linear maps and regularity

It is well known that every real linear map $\Lambda : X \to Y$ between complex spaces may be uniquely decomposed into the sum

$$\Lambda = L_\Lambda + A_\Lambda \tag{4.1}$$

of a complex linear and a complex anti–linear part, respectively. Straightforward computations lead to

$$L_\Lambda(x) = \frac{\Lambda(x) - i\Lambda(ix)}{2} \qquad\qquad A_\Lambda(x) = \frac{\Lambda(x) + i\Lambda(ix)}{2}\ .$$

If $\Lambda$ is the real differential of a function $f : X \to Y$, the vanishing of one or the other component in a open subset $U \subset X$ decides the holomorphy type of $f$ in $U$: by expressing it in a complex basis of $X$, one finds the classical Cauchy–Riemann equations of the complex analysis.

When $X$ and $Y$ are quaternionic spaces, the regularity of $f$ has been introduced by adapting the Cauchy–Riemann equations to the increased number of imaginary units: see [6] or [23] for the one–dimensional case, and [8] for the general

case. The question addressed in this section is whether these equations originate from a canonical decomposition of $df$, as for the complex case: the answer is in the positive, and completes the results in [22] and [8].
The failure of the word–by–word translation of (4.1) to the quaternionic case is well known in literature, the most popular explanation being a dimensional fact: when $X = Y = \mathbb{H}$, the identity and the conjugation alone cannot span the four dimensional quaternionic space $\mathrm{Lin}^{\mathbb{R}}(\mathbb{H})$. However, Proposition 3.1 says that there is a more relevant obstruction, which prevents to complete them to a quaternionic basis of $\mathrm{Lin}^{\mathbb{R}}(X; Y)$: the linearity with respect to the identity requires that $X$ and $Y$ have the same chirality, while the linearity with respect to the conjugation needs different chiralities. In fact, the true core of all the story is that these two notions are no longer complementary as in the complex case, but play exactly the same role in a different context of chiralities: roughly speaking, the main result of this section states that regular maps are those which are as far as possible from them.
The idea is to decompose $\mathrm{Lin}^{\mathbb{R}}(X; Y)$ along some suitable classes of quaternionic linear maps. To guess how to choose the classes, come back to the complex case and look at the notions of complex linearity, which are involved in (4.1). Using the terminology of Section 3 also for the complex case, it is clear that the component $L_\Lambda$ is complex linear with respect to the identity $\mathrm{id} : \mathbb{C} \to \mathbb{C}$, while $A_\Lambda$ is complex linear with respect to the conjugation $\mathrm{cj} : \mathbb{C} \to \mathbb{C}$. The identity and conjugation are the only real algebra automorphisms of $\mathbb{C}$ and, what is really important here, they are *orthogonal in a complex sense*. Indeed

$$\big(\mathrm{id} \,|\, \mathrm{cj}\big) \;=\; \frac{1 + i^2}{2} \;=\; 0$$

where the scalar product is the standard complex one on $\mathrm{Lin}^{\mathbb{R}}(\mathbb{C})$, recalled in Remark 5.2. The one–dimensional quaternionic case $X = Y = \mathbb{H}$ behaves similarly: as explained in Section 2, every set of four orthogonal automorphisms spans all $\mathrm{Lin}^{\mathbb{R}}(\mathbb{H})$ over the quaternions. The next proposition extends this fact to the general case, moreover providing an explicit computational rule.

**Proposition 4.1.** *Assume that the automorphisms $\omega_0, \omega_1, \omega_2, \omega_3$ are quaternionic orthogonal. Then every $\Lambda \in \mathrm{Lin}^{\mathbb{R}}(X; Y)$ can be uniquely decomposed as*

$$\Lambda \;=\; \Lambda_0 + \Lambda_1 + \Lambda_2 + \Lambda_3$$

*where $\Lambda_h \in \mathrm{Lin}^{\mathbb{H}}_{\varphi_h}(X; Y)$ for every $h$, and*

$$\varphi_h = \omega_h \qquad \text{or} \qquad \varphi_h = \overline{\omega}_h$$

*depending on whether $X$ and $Y$ have or have not the same chirality. Moreover, for every $h$ one has*

$$\Lambda_h(x) \;=\; \frac{1}{4} \sum_{k=0}^{3} \overline{\varphi}_h(e_k) * \Lambda(e_k * x) \tag{4.2}$$

*with $e_0, e_1, e_2, e_3$ any real orthornomal basis of $\mathbb{H}$.*

A special case of this result was already obtained in [8], without any reference to orthogonality and for automorphisms $\varphi_h$'s which are coupled with the $e_h$'s in the

following sense: $\omega_h$ is the inner automorphism associated to $e_h$, with the additional assumption that $e_0 = 1$.
These choices certainly produce simpler coefficients in (4.2), but hide the invariance properties of the decomposition. Decoupling automorphisms of $\mathbb{H}$ and bases of $\mathbb{H}$ makes clear that, for instance, the second ones do not affect formula (4.2): $\Lambda_h$ does not change when the $e_0, e_1, e_2, e_3$ vary. A much more relevant invariance property is pointed out in Corollary 4.2.

*Proof.* Assume first that a decomposition exists, and prove that (4.2) follows. To this aim, evaluate the decomposition at the point $e_k * x$ obtaining

$$\Lambda(e_k * x) = \sum_{n=0}^{3} \varphi_n(e_k) * \Lambda_n(x) \ .$$

Thus, computing the right hand side of (4.2) one has

$$\begin{aligned}
\textstyle\sum_{k=0}^{3} \overline{\varphi}_h(e_k) * \Lambda(e_k * x) &= \sum_{n=0}^{3} \frac{1}{4} \sum_{k=0}^{3} \overline{\varphi}_h(e_k) * \{\varphi_n(e_k) * \Lambda_n(x)\} \\
&= \textstyle\sum_{n=0}^{3} \begin{cases} (\varphi_h \mid \varphi_n) * \Lambda_n(x) \\ (\varphi_n \mid \varphi_h) * \Lambda_n(x) \end{cases}
\end{aligned}$$

depending on the chirality of $Y$. The orthogonality of the reference morphisms then yields formula (4.2). It remains to prove that this formula really defines a quaternionic linear map, with respect to $\varphi_h$. To this aim, begin by noticing that

$$\begin{aligned}
\Lambda_h(e_j * x) &= \frac{1}{4} \sum_{k=0}^{3} \overline{\varphi_h}(e_k) * \Lambda\big(e_k * (e_j * x)\big) \\
&= \frac{1}{4} \sum_{k=0}^{3} \varphi_h(e_j) * \Big\{\overline{\varphi_h}(e_j) * \big\{\overline{\varphi_h}(e_k) * \Lambda\big(e_k * (e_j * x)\big)\big\}\Big\} \\
&= \varphi_h(e_j) * \frac{1}{4} \sum_{k=0}^{3} \begin{cases} \overline{\varphi_h}(e_k e_j) * \Lambda\big((e_k e_j) * x\big) \\ \overline{\varphi_h}(e_j e_k) * \Lambda\big((e_j e_k) * x\big) \end{cases}
\end{aligned}$$

where the alternative depends on the chirality of $X$. Since, when $k$ varies, both the $e_k e_j$'s and the $e_j e_k$'s describe a real orthonormal basis of $\mathbb{H}$, one finds

$$\Lambda_h(e_j * x) = \varphi_h(e_j) * \frac{1}{4} \sum_{k=0}^{3} \overline{\varphi}_h(e_k) * \Lambda(e_k * x) = \varphi_h(e_j) * \Lambda_h(x) \ .$$

This is true for every $j$, and the claim follows from the manifest additivity and real homogeneity of $\Lambda_h$. ∎

When $X = Y = \mathbb{H}$ and the quaternionic structure of $\mathrm{Lin}^{\mathbb{R}}(\mathbb{H})$ is decided by the chirality context, with the rules of the previous section, the above decomposition becomes orthogonal in the quaternionic sense, namely it satisfies

$$\big(\Lambda_h \mid \Lambda_k\big) = 0$$

for every $h \neq k$: see Lemma 3.5. Here, of course, the chirality of the scalar product is that of $\mathrm{Lin}^{\mathbb{R}}(\mathbb{H})$. In the general case, the orthogonality survives at a formal level only, since so are the associated automorphisms. The decomposition, however, is more orthogonal than it seems: formula (4.2) shows that each component $\Lambda_h$ only depends on the choice of $\omega_h$, in the sense that it is not affected by any change of the $\omega_k$'s with $k \neq j$. This trivial fact proves the following important invariance property.

**Corollary 4.2.** *If* $\omega_0 = \mathrm{id}$ *in Proposition 4.1, then the two components*

$$\mathcal{L}_\Lambda = \Lambda_0 \qquad \text{and} \qquad \mathcal{R}_\Lambda = \Lambda_1 + \Lambda_2 + \Lambda_3$$

*do not change when the imaginary orthogonal automorphisms* $\omega_1, \omega_2, \omega_3$ *vary.*

The two components above give rise to complementary projections in $\mathrm{Lin}^{\mathbb{R}}(X;Y)$, inasmuch

$$\mathcal{L}_{\mathcal{L}_\Lambda} = \mathcal{L}_\Lambda \qquad\qquad \mathcal{L}_{\mathcal{R}_\Lambda} = \mathcal{R}_{\mathcal{L}_\Lambda} = 0 \qquad\qquad \mathcal{R}_{\mathcal{R}_\Lambda} = \mathcal{R}_\Lambda$$

for every $\Lambda$. The resulting decomposition

$$\Lambda \;=\; \mathcal{L}_\Lambda + \mathcal{R}_\Lambda \tag{4.3}$$

deserves the name of *canonical decomposition* of $\Lambda$: as it will be clear in a while, it plays for the quaternionic analysis exactly the same role of (4.1) for the complex analysis. The two components $\mathcal{L}_\Lambda$ and $\mathcal{R}_\Lambda$ will be referred to as the *linear* and the *regular* part of $\Lambda$, respectively. The first one is indeed quaternionic linear with respect to the identity or the conjugation, depending on the chirality context. The second one is made by imaginary quaternionic linear maps, though it is not itself quaternionic linear in general: its name clearly reflects the following idea.

**Definition 4.3.** *A map* $\Lambda$ *is regular when* $\mathcal{L}_\Lambda = 0$.

In a similar way, a map satisfying $\mathcal{R}_\Lambda = 0$ is called a *linear map*, without any further qualifier like real or quaternionic: this notion corresponds to the standard quaternionic linearity. The classes of the linear and the regular maps are denoted by the symbols $\mathcal{L}(X;Y)$ and $\mathcal{R}(X;Y)$ respectively, so that the canonical decomposition yields

$$\mathrm{Lin}^{\mathbb{R}}(X;Y) = \mathcal{L}(X;Y) \oplus \mathcal{R}(X;Y)\,. \tag{4.4}$$

Both the components are real subspaces of $\mathrm{Lin}^{\mathbb{R}}(X;Y)$, and become quaternionic subspaces as soon as $Y$ has a double quaternionic structure, and the chirality rules of the previous section are respected. This is of course the case when $Y = \mathbb{H}$. When also $X = \mathbb{H}$,

$$\big(\mathcal{L}_\Lambda \,|\, \mathcal{R}_\Lambda\big) \;=\; 0$$

for every $\Lambda$, namely the decomposition (4.4) becomes quaternionic orthogonal. Since $\mathcal{L}(\mathbb{H})$ is the quaternionic span of either the identity or the conjugation, depending on the chirality context, the regularity takes the quite suggestive, purely metric form given in the next corollary: this result fully accounts for the opening claim of the section, saying that regularity is as far as possible from the standard quaternionic linearity.

**Corollary 4.4.** *A map $\Lambda \in \mathrm{Lin}^{\mathbb{R}}(\mathbb{H})$ is regular if and only if*

$$\big(\Lambda \mid \varphi_0\big) = 0 \tag{4.5}$$

*where either $\varphi_0 = \mathrm{id}$ or $\varphi_0 = \mathrm{cj}$, depending on the chirality context.*

Consider now a possible nonlinear function $f : U \subset X \to Y$ between quaternionic Banach spaces, where $U$ is an open subset of $X$: the regularity of $f$ in $U$ amounts to require that $f$ is real differentiable in $U$ and that

$$df(x) \in \mathcal{R}(X;Y) \qquad \forall x \in U\,.$$

The class of these maps will be denoted by $\mathrm{Reg}(U, Y)$. Of course, in order for this definition to be admissible, one must show that the equations (1.1)–(1.2) are recovered in the one–dimensional setting. Doing that, one immediately realizes a fundamental fact: *the choice among the possible Cauchy–Riemann equations is not just a matter of taste, but is forced by the chiralities of $X$ and $Y$.*
The reason may be easily seen from (4.5), but a more convincing explanation may be obtained by writing the differential version of the canonical decomposition. To begin with, assume that $X = \mathbb{H}$ and notice that $\Lambda_h(v) = \varphi_h(v) * \Lambda_h(1)$ by quaternionic linearity. Assume now that $\Lambda = df(x)$ and write $\partial_v f(x)$ for the directional derivative of $f$ at $x$ along $v$, namely

$$\partial_v f(x) \;=\; df(x)\cdot v\,.$$

Casting all that into (4.2) then yields

$$[df(x)]_h \cdot v = \varphi_h(v) * \Big\{\frac{1}{4}\sum_{k=0}^{3} \overline{\varphi}_h(e_k) * \partial_{e_k} f(x)\Big\}$$

and introducing the differential operators

$$\frac{\partial f}{\partial \varphi_h}(x) \;=\; \frac{1}{4}\sum_{k=0}^{3} \overline{\varphi}_h(e_k) * \partial_{e_k} f(x) \tag{4.6}$$

the canonical decomposition takes the form

$$df(x)\cdot v \;=\; \varphi_0(v) * \frac{\partial f}{\partial \varphi_0}(x) \;+\; \sum_{h=1}^{3} \varphi_h(v) * \frac{\partial f}{\partial \varphi_h}(x)\,.$$

To write it in a similar way to the complex analogous (1.3), again two steps are needed. Thinking of the differential operator (4.6) as to the analogous of $\partial f/\partial z$ in (1.3), one would like to know what to do with $dz$: the correct translation reveals to be the differential form

$$d\varphi_h(x)\cdot v = \varphi_h(v)$$

which is constant in $U$. The second step is to set also $Y = \mathbb{H}$, and to make use of the second quaternionic action $\diamond$ on it, that having the opposite chirality with respect to $*$. This leads to the differential decomposition

$$df \;=\; \frac{\partial f}{\partial \varphi_0} \diamond d\varphi_0 \;+\; \sum_{h=1}^{3} \frac{\partial f}{\partial \varphi_h} \diamond d\varphi_h\,. \tag{4.7}$$

The regularity of $f$ in $U$, as defined in this section, corresponds to the vanishing of the coefficient of the first term in (4.7), namely to the fact that the differential condition

$$\sum_{k=0}^{3} \overline{\varphi}_0(e_k) * \partial_{e_k} f = 0$$

is satisfied in all $U$. By choosing the chirality context, one is now able to decide if $*$ is a left or a right multiplication, and if $\varphi_0$ is the identity or the conjugation: it is not difficult to check that the two first equations in (1.1)–(1.2) correspond to the case of equal chiralities, while the last two correspond to different chiralities.

*Remark* 4.5. The decomposition (4.7) must be compared with the analogous one in [22], which refers implicitly to the case of equal, left chiralities and writes as

$$df \;=\; dq\,\partial f + dq^1\,\partial^1 f + dq^2\,\partial^2 f + dq^3\,\partial^3 f\,. \tag{4.8}$$

In this formula $dq = d\varphi_0$ and

$$\partial f = \frac{\partial f}{\partial \varphi_0} = \frac{1}{4}\sum_{h=0}^{3} \overline{e_h}\,\partial_{e_h} f$$

while $dq^k = -e_k\,dq\,e_k = d\varphi_k$ and $\partial^k f = -e_k\,\partial f\,e_k$ for $k \geq 1$. Thus, the differential forms are the same and in the same position of (4.7), while the differential coefficients are different for $k \geq 1$. The point is that (4.8) is correct for a function $f$ with values in $\mathbb{R} \subset \mathbb{H}$, but false in the general case. To see why, test it on the map $f(t+u) = u$, where $t \in \mathbb{R}$ and $u \in \mathbb{R}^3$. It is $\partial f = 3/4$ and, rewriting (4.8) as

$$df \;=\; dq\,\partial f - e_1\,dq\,\partial f\,e_1 - e_2\,dq\,\partial f\,e_2 - e_3\,dq\,\partial f\,e_3$$

(see also [22]) it is not difficult to check that the right hand side gives $3\,dt$. Thus, the equality is not satisfied. It is remarkable that none of the other conclusions in [22] is affected by this mistake. Finally, it should be noticed that the algebraic nature of the decomposition is not investigated in that paper.

Coming back to the general framework, the effect of the chirality context on the notion of regularity is summarized by the following table

$4\,\mathcal{L}_\Lambda(x) \;=$

| $X$ left | $X$ right | |
|---|---|---|
| $\sum_{k=0}^{3} \bar{e}_k\Lambda(e_k x)$ | $\sum_{k=0}^{3} e_k\Lambda(x e_k)$ | $Y$ left |
| $\sum_{k=0}^{3} \Lambda(e_k x)e_k$ | $\sum_{k=0}^{3} \Lambda(x e_k)\bar{e}_k$ | $Y$ right |

where, for the computations, standard bases have been used, which are moreover related via the inner representation. This table provides an equivalent definition of regularity, which is independent of any decomposition or linearity argument: this was indeed the choice made in [8].

*Remark* 4.6. The reader might wonder if, by completing the previous table, all the Cauchy–Riemann equations can be obtained in a single chirality context. For instance, when $X = \mathbb{H}$ is a right space and $Y$ is a left space, one may also think of (1.1) as to the vanishing at $x = 1$ of

$$\mathcal{A}_\Lambda(x) \;=\; \frac{1}{4} \sum_{k=0}^{3} e_k \Lambda(e_k x) \,,$$

although this map does not originate from a canonical decomposition of $\Lambda$. The point is, however, that the map $\mathcal{A}_\Lambda$ may vanish somewhere without being trivial: in [8] it was already shown that the kernel of $\mathcal{A}_\Lambda$ is not, in general, neither a left nor a right quaternionic subspace of $\mathbb{H}$. Of course, the quaternionic linearity of $\mathcal{L}_\Lambda$ guarantees that this cannot happen for the entries of the table.

Whatever the definition is, (4.3) expresses the fact that every real linear map has a formally orthogonal decomposition into a linear and a regular part. The further decomposition of the regular part determines the algebraic nature of the regularity itself: the next lemma is the first step in this direction.

**Lemma 4.7.** *A nontrivial quaternionic linear map is regular if and only if it is imaginary.*

*Proof.* Assume that $\Lambda \in \mathrm{Lin}^{\mathbb{H}}_{\varphi}(X;Y)$ and, with the notations of Corollary 4.2, compute its linear part as

$$\mathcal{L}_\Lambda(x) = \frac{1}{4} \sum_{k=0}^{3} \overline{\varphi}_0(e_k) * \Lambda(e_k * x) = \frac{1}{4} \sum_{k=0}^{3} \overline{\varphi}_0(e_k) * \{\varphi(e_k) * \Lambda(x)\} \,.$$

This yields

$$\mathcal{L}_\Lambda(x) = \begin{cases} (\varphi_0 \mid \varphi) * \Lambda(x) \\ (\varphi \mid \varphi_0) * \Lambda(x) \end{cases}$$

depending on whether $Y$ has the same chirality of $X$ or not. The thesis follows from the fact that $\varphi_0$ is either the identity or the conjugation, depending on the chirality context. ■

As a consequence, the sum of imaginary quaternionic linear maps is regular. On the other hand, Corollary 4.2 says that every regular maps is the sum of three orthogonal imaginary quaternionic linear maps. By forgetting orthogonality, one has the following purely algebraic characterization.

**Corollary 4.8.** *A real linear map is regular if and only if is the sum of imaginary quaternionic linear maps.*

A couple of remarks, about what the corollary doesn't say, are maybe worth. First of all, it does not say that the decomposition into imaginary quaternionic linear maps is unique: there are as many of them, as the real orthonormal bases of $\mathbb{R}^3$. Secondly, and more relevant, this is not the only way to obtain a regular map. For instance, it is not difficult to check that

$$\vartheta_{1+i}(x)\,\frac{1-i}{2} + \vartheta_{1-i}(x)\,\frac{1+i}{2} \;=\; \vartheta_i(x)$$

holds for every $x \in \mathbb{H}$: when working between left spaces, this shows that the regular map $\vartheta_i$ may be written as the sum of two nonimaginary quaternionic linear maps.

All is ready to draw the parallel between the regular maps and the imaginary quaternions, mentioned in the Introduction, collecting all the indications disseminated in the paper. In the one–dimensional case, one knows that

$$\mathrm{Lin}^{\mathbb{R}}(\mathbb{H}) = \mathcal{L}(\mathbb{H}) \oplus \mathcal{R}(\mathbb{H})$$

is an orthogonal decomposition, with respect to a natural quaternionic scalar product, which depends on the chirality context. From a purely dimensional point of view, this recalls the classical decomposition

$$\mathbb{H} = \mathbb{R} \oplus \mathbb{R}^3$$

which is also orthogonal, but in a real sense. The inner representation $\vartheta$ maps the real quaternions into the identity, and $\mathbb{R}^3$ into the imaginary automorphisms, which are special regular maps (in the appropriate chirality context). In fact, like the imaginary units in $\mathbb{R}^3$, the imaginary automorphisms span all the regular maps, over the quaternions, and the formulas (2.11) confirm the analogy: not surprisingly, the role of the product in $\mathbb{H}$ is played by the composition in $\mathrm{Lin}^{\mathbb{R}}(\mathbb{H})$. An even tighter analogy may be obtained by studying the behavior of the regularity under composition: it is well known that regularity is not preserved, in general, and the aim here is to understand the algebraic reason of that. Since this is better seen in general space, consider two maps

$$\Lambda \in \mathrm{Lin}^{\mathbb{R}}(X;Y) \qquad \Gamma \in \mathrm{Lin}^{\mathbb{R}}(Y;Z) \tag{4.9}$$

with no restrictions on the chirality contexts, apart from the obvious fact that $Y$, even when it may be seen as a quaternionic spaces in many different ways, has the same quaternionic structure in the two cases. Write the canonical decompositions

$$\Lambda = \mathcal{L}_\Lambda + \mathcal{R}_\Lambda \qquad \Gamma = \mathcal{L}_\Gamma + \mathcal{R}_\Gamma \tag{4.10}$$

and try to determine how the linear and the regular parts transform under composition. There's no doubts that

$$\mathcal{L}_\Gamma \circ \mathcal{L}_\Lambda \in \mathcal{L}(X;Z) \qquad \mathcal{L}_\Gamma \circ \mathcal{R}_\Lambda + \mathcal{R}_\Gamma \circ \mathcal{L}_\Lambda \in \mathcal{R}(X;Z) \tag{4.11}$$

so that they contribute to the linear and the regular parts of $\Gamma \circ \Lambda$, respectively. This follows from the composition rule (3.3), by looking at the respective reference morphisms: indeed, the composition with the identity or with the conjugation does not change the nature of an imaginary automorphism. The true question concerns the algebraic nature of $\mathcal{R}_\Gamma \circ \mathcal{R}_\Lambda$. To study it, begin by introducing the following two operations

$$\begin{aligned}
\mathcal{R}_\Gamma \cdot \mathcal{R}_\Lambda &= \Gamma_1 \circ \Lambda_1 + \Gamma_2 \circ \Lambda_2 + \Gamma_3 \circ \Lambda_3 \\
\mathcal{R}_\Gamma \times \mathcal{R}_\Lambda &= \Gamma_2 \circ \Lambda_3 + \Gamma_3 \circ \Lambda_2 + \Gamma_1 \circ \Lambda_3 + \Gamma_3 \circ \Lambda_1 + \\
&\quad \Gamma_1 \circ \Lambda_2 + \Gamma_2 \circ \Lambda_1
\end{aligned}$$

where, of course, the decomposition of the regular parts

$$\mathcal{R}_\Lambda = \Lambda_1 + \Lambda_2 + \Lambda_3 \qquad \mathcal{R}_\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 \tag{4.12}$$

is that provided by Corollary 4.2. Though

$$\mathcal{R}_\Gamma \circ \mathcal{R}_\Lambda \;=\; \mathcal{R}_\Gamma \cdot \mathcal{R}_\Lambda + \mathcal{R}_\Gamma \times \mathcal{R}_\Lambda \tag{4.13}$$

by construction, it should be clear that the right hand side is not made by canonical objects, in general: next lemma specifies the missing ingredient for that. To state it, one needs to compare the quaternionic linearity of maps which belong to different spaces, as in (4.9), to give a sense to statements like: $\Lambda$ has the same reference morphisms of $\Gamma$, or their reference morphisms are orthogonal, and so on. The meaning is obvious when the relative chirality context of the two pairs $X, Y$ and $Y, Z$ is the same, namely: either the chiralities are the same for both the pairs, or they are different for both of them. When the relative chirality contexts are different, having the same reference morphism means that: if $\Lambda$ is quaternionic linear with respect to $\varphi$, then $\Gamma$ is with respect to $\overline{\varphi}$. It is not difficult to guess that everything works fine in the computations, because the conjugation commutes with all the automorphisms: see (2.3) in Section 2.

**Lemma 4.9.** *Assume that the two decompositions in (4.12) refer to the same standard basis. Then $\mathcal{R}_\Gamma \cdot \mathcal{R}_\Lambda$ and $\mathcal{R}_\Gamma \times \mathcal{R}_\Lambda$ are canonical objects and satisfy*

$$\mathcal{R}_\Gamma \cdot \mathcal{R}_\Lambda \;=\; \mathcal{L}_{\mathcal{R}_\Gamma \circ \mathcal{R}_\Lambda} \qquad\qquad \mathcal{R}_\Gamma \times \mathcal{R}_\Lambda \;=\; \mathcal{R}_{\mathcal{R}_\Gamma \circ \mathcal{R}_\Lambda} \,.$$

In particular, when $\Lambda$ and $\Gamma$ are already regular maps, their composition $\Gamma \circ \Lambda$ is regular if and only if

$$\Gamma_1 \circ \Lambda_1 + \Gamma_2 \circ \Lambda_2 + \Gamma_3 \circ \Lambda_3 = 0 \,.$$

This is a kind of algebraic orthogonality condition which, for instance, is satisfied when $\Lambda$ and $\Gamma$ are mutually orthogonal in the usual formal sense: for each index $k$, either $\Lambda_k = 0$ or $\Gamma_k = 0$.

*Proof.* The first statement follows from the second. The assumption says that both the decompositions in (4.12) are shadowed by the same standard basis of automorphisms: denote it by $\mathrm{id}, \omega_1, \omega_2, \omega_3$. The composition rules of these automorphisms are stated in (2.11) and yield

$$\mathcal{R}_\Gamma \cdot \mathcal{R}_\Lambda \in \mathcal{L}(X;Y) \qquad\qquad \mathcal{R}_\Gamma \times \mathcal{R}_\Lambda \in \mathcal{R}(X;Y) \,.$$

Thus (4.13) must be the canonical decomposition of $\mathcal{R}_\Gamma \circ \mathcal{R}_\Lambda$. ∎

The lemma yields the composition rule

$$\begin{aligned} \Gamma \circ \Lambda \;=\; & \left\{ \mathcal{L}_\Gamma \circ \mathcal{L}_\Lambda + \mathcal{R}_\Gamma \cdot \mathcal{R}_\Lambda \right\} + \\ & \left\{ \mathcal{L}_\Gamma \circ \mathcal{R}_\Lambda + \mathcal{R}_\Gamma \circ \mathcal{L}_\Lambda + \mathcal{R}_\Gamma \times \mathcal{R}_\Lambda \right\} \end{aligned} \tag{4.14}$$

where the two parentheses define the linear and the regular parts of $\Gamma \circ \Lambda$, respectively: the formal analogy with the product of quaternions is manifest, and

completes the parallel between regular maps and imaginary quaternions.
To think of the regular maps as imaginary quaternions, is especially convenient when trying to guess further properties of the former. For instance, the invariance of $\mathbb{R}^3$ under the (inner) automorphisms suggests that regularity is preserved by some special composition with quaternionic linear maps. Consider indeed, in addition to the $\Lambda$ and $\Gamma$ used until now, a third map

$$\Pi \in \mathrm{Lin}^{\mathbb{R}}(W; X)$$

acting before $\Lambda$. The following result holds.

**Lemma 4.10.** *Assume that $\Pi$ and $\Gamma$ are quaternionic linear maps, and that the corresponding reference morphisms are one the inverse of the other. If $\Lambda$ is regular then also $\Gamma \circ \Lambda \circ \Pi$ is.*

The lemma extends a well known, one-dimensional result by Sudbery [23], where quaternionic linearity is replaced by conformality: indeed, in Section 6 it will be shown that the two notions coincide in $\mathbb{H}$.

*Proof.* The assumption says that the reference morphisms of $\Pi$ and $\Gamma$ are shadowed by an automorphism $\omega$ and its inverse $\omega^{-1}$, respectively. Write $\omega = \vartheta_q$ for some quaternion $q \neq 0$. The claim follows from the algebraic characterization of regularity, as soon as one notices that

$$\vartheta_q \circ \vartheta_g \circ \left(\vartheta_q\right)^{-1} = \vartheta_{qgq^{-1}}$$

and that $qgq^{-1}$ is imaginary when $g$ is. ∎

## 5 Quaternionic size of regular maps and complex linearity

According to the previous section, every regular real linear map is the sum of three imaginary and orthogonal quaternionic linear maps. However, how many of them are really needed may depend on the choice of the reference morphisms: this quantity deserves a name.

**Definition 5.1.** *The quaternionic size $s\{\Lambda\}$ of a map $\Lambda \in \mathcal{R}(X; Y)$ is the minimal number of nonzero and mutually orthogonal imaginary quaternionic linear maps, whose sum is $\Lambda$.*

By construction

$$0 \leq s\{\Lambda\} \leq 3$$

for every regular $\Lambda$. This number is zero only when $\Lambda = 0$, and it is one for all the nontrivial imaginary quaternionic linear maps. The main point is to understand size two maps: the aim hereafter is to show that they are a quite special type of *complex linear* maps.
To set up the problem, one has first to look at a quaternionic space as a complex space. In principle, this is absolutely trivial: as it was already done for $\mathbb{R}$, it is

sufficient to look at $\mathbb{C}$ as to a subset of $\mathbb{H}$, and then to restrict the scalar multiplication. However, there is a relevant difference between the two cases: while $\mathbb{R}$ is univocally determined as the center of $\mathbb{H}$, the insertion of $\mathbb{C}$ in $\mathbb{H}$ depends on arbitrary choices. For instance, every unitary $g \in \mathbb{R}^3$ defines a field morphism by the rule

$$z = t + s\,i \in \mathbb{C} \;\mapsto t + s\,g \in \mathbb{H} \tag{5.1}$$

where $t, s \in \mathbb{R}$. In fact, there are many other ways to insert $\mathbb{C}$ into $\mathbb{H}$, all of them being discontinuous: the continuity is equivalent to fixing $\mathbb{R}$ and produces real algebra morphisms. This way, one selects exactly the morphisms defined by (5.1): they are the only ones of interest here.

Consider now two quaternionic spaces $X$ and $Y$, and let $g$ and $h$ be the imaginary units which define their complex structures, respectively. According to the usual definition, a map $\Lambda : X \to Y$ is said *complex linear of type* $(g, h)$ when it is additive and

$$\Lambda((t + sg) * x) = (t + sh) * \Lambda(x)$$

for all $x \in X$ and all $t, s \in \mathbb{R}$. Of course, this is the same to say that $\Lambda$ is real linear and satisfies

$$\Lambda(g * x) = h * \Lambda(x)$$

for every $x \in X$. The class of such $\Lambda$'s is denoted by $\mathrm{Lin}^{\mathbb{C}}_{gh}(X; Y)$. Notice that also the standard complex anti–linearity is included in this scheme: it corresponds to the complex linearity of the type $(g, -h)$. This yields the classical decomposition

$$\mathrm{Lin}^{\mathbb{R}}(X; Y) = \mathrm{Lin}^{\mathbb{C}}_{gh}(X; Y) \oplus \mathrm{Lin}^{\mathbb{C}}_{g(-h)}(X; Y)\ . \tag{5.2}$$

The complex linear maps in the quaternionic framework enjoy all the usual properties of complex analysis, when settled in the correct context. For instance, the fact that the composition of complex linear maps is again complex linear must be now read in the following way: $\Lambda \in \mathrm{Lin}^{\mathbb{C}}_{gh}(X; Y)$ and $\Gamma \in \mathrm{Lin}^{\mathbb{C}}_{hl}(Y; Z)$ imply

$$\Gamma \circ \Lambda \in \mathrm{Lin}^{\mathbb{C}}_{gl}(X; Z)\ . \tag{5.3}$$

All the spaces considered above are real subspaces of $\mathrm{Lin}^{\mathbb{R}}(X; Y)$: in fact, under the usual assumptions of Section 3, all of them are also quaternionic subspaces.

*Remark* 5.2. It should be noticed that no complex structures has been considered on the space $\mathrm{Lin}^{\mathbb{C}}_{gh}(X; Y)$. The most natural of them certainly is the pointwise operation associated to $*$ in $Y$. However, this choice is not compatible with the quaternionic structure of $\mathrm{Lin}^{\mathbb{R}}(X; Y)$, as introduced in Section 3 by means of a second scalar multiplication $\diamond$ on $Y$ (if any, of course): this latter structure is the only one of interest here.

To make clear how complex linearity is related to quaternionic linearity, begin by wondering if the type of complex linearity of a given map is uniquely defined. In general, the answer is in the negative. For instance, it is not difficult to check that

$$\mathrm{Lin}^{\mathbb{H}}_{\varphi}(X; Y) \;\subset\; \mathrm{Lin}^{\mathbb{C}}_{g\,\varphi(g)}(X; Y) \tag{5.4}$$

for every imaginary unit $g$. The problem is whether this is the only indeterminacy or not: the answer is provided hereafter, by looking at the nature of a map

$$\Lambda \in \mathrm{Lin}^{\mathbb{C}}_{gh}(X;Y) \cap \mathrm{Lin}^{\mathbb{C}}_{g'h'}(X;Y) \tag{5.5}$$

which is complex linear with respect to different complex structures. In [21], these type of maps are called *quaternionic maps*: it will be clear in a while that they are exactly the quaternionic linear maps, introduced in Section 3. Next lemma specifies the compatibility conditions, for such a $\Lambda$ to really exist.

**Lemma 5.3.** *If $\Lambda \neq 0$ satisfies (5.5) then*

$$g \cdot g' = h \cdot h' \tag{5.6}$$

For instance, since everything is unitary, $g' = g$ yields $h' = h$ while $g' = -g$ yields $h' = -h$.

*Proof.* From (2.2) one has $-2\, u \cdot v = uv + vu$, for every pair of imaginary quaternions $u$ and $v$. Then by using sequentially the two types of complex linearities of $\Lambda$, together with its real linearity, one has

$$\begin{aligned}&-2(g \cdot g') * \Lambda(x) = \Lambda\big(-2(g \cdot g') * x\big) = \Lambda\big((gg' + g'g) * x\big)\\ &= (hh' + h'h) * \Lambda(x) = -2(h \cdot h') * \Lambda(x)\end{aligned}$$

whatever the chiralities of $X$ and $Y$ are. The claim follows by taking $x$ such that $\Lambda(x) \neq 0$. ∎

It is clear that, in some cases, a second complex linearity type does not add any new information. For instance, this is the case when $g' = -g$ and $h' = -h$, since

$$\mathrm{Lin}^{\mathbb{C}}_{(-g)(-h)}(X;Y) = \mathrm{Lin}^{\mathbb{C}}_{gh}(X;Y)$$

due to real linearity of the involved maps. Next lemma says what happens in all the other cases.

**Proposition 5.4.** *Assume that $\Lambda$ satisfies (5.5) with $g \times g' \neq 0$. Then $\Lambda$ is quaternionic linear.*

Notice that $g \times g' = 0$ if and only if $g' = \pm g$. Thus, as a consequence of (5.6), the only situations where the hypothesis and then the conclusion of the lemma do not hold are: the trivial one, namely $g' = g$ and $h' = h$, and the one with the reversed signs, namely $g' = -g$ and $h' = -h$. As already said, the two choices produce the same class of complex linear functions. Hence, after identifying the corresponding pairs of complex structures, the lemma says that: a map, which is not quaternionic linear, can be complex linear with respect to a single choice of the pair of complex structures. In this sense, the result is the complex analogous of Lemma 3.2, about the uniqueness of the reference morphism for a quaternionic linear map.

*Proof.* As in the proof of the previous lemma, since $2\,u \times v = uv - vu$ for every imaginary $u$ and $v$, one has

$$\Lambda\big((g \times g') * x\big) = (h \times h') * \Lambda(x) \qquad \text{or} \qquad \Lambda\big((g \times g') * x\big) = -(h \times h') * \Lambda(x)$$

for every $x$, according to whether $X$ and $Y$ have or have not the same chirality. Now, by construction, the set $\{1, g, g', g \times g'\}$ is a (possibly non orthogonal) real basis of $\mathbb{H}$. Consider the unique $\varphi \in \mathrm{Lin}^{\mathbb{R}}(\mathbb{H})$ defined by

$$\varphi(1) = 1 \qquad \varphi(g) = h \qquad \varphi(g') = h' \qquad \varphi(g \times g') = \pm\, h \times h'\ ,$$

where the choice of the last sign depends once more on the chirality context. The real linearity of $\Lambda$ then allows to conclude that

$$\Lambda(q * x) = \varphi(q) * \Lambda(x)$$

for every quaternion $q$ and every $x$. ∎

*Remark* 5.5. Apparently, condition (5.6) does not play any role in the previous proposition. However, if (5.5) is satisfied and (5.6) is not, then $\Lambda$ must be trivial, due to Lemma 5.3: in this case, the proposition is true but trivial. When $\Lambda$ is not trivial, condition (5.6) must hold, and it is not difficult to check that it implies that $\varphi$ acts on $\mathbb{R}^3$ as a rotation, with positive or negative determinant, depending on the chirality context. Hence, $\varphi$ is either an automorphism or an anti–automorphism of $\mathbb{H}$: this is the same conclusion of Proposition 3.1.

After this premise, the characterization of complex linearity by the quaternionic size may really start.

**Proposition 5.6.** *The sum of two quaternionic linear maps (between the same spaces) is a complex linear map.*

*Proof.* Let $\Lambda$ and $\Gamma$ be quaternionic linear with respect to $\varphi$ and $\psi$ respectively. Due to (5.4), to prove the claim is enough to find an imaginary unit $g$ such that $\varphi(g) = \psi(g)$. Assume by definiteness that $X$ and $Y$ have the same chirality, and write $\varphi = \vartheta_p$ and $\psi = \vartheta_q$ for some suitable nonzero $p, q \in \mathbb{H}$. Then the previous condition becomes $p\,gp^{-1} = qgq^{-1}$ which may be better seen as

$$(q^{-1}p)g(q^{-1}p)^{-1} = g\ .$$

Now Lemma 2.2 applies to show that

$$g = \frac{\mathrm{Im}\ (q^{-1}p)}{|\mathrm{Im}\ (q^{-1}p)|}$$

does perfectly the job. ∎

The next step is to invert the implication in Proposition 5.6. To this aim, one has to reverse the perspective in (5.4), by first fixing the type $(g, h)$ of complex linearity which is interested in, and then looking for the reference morphisms $\varphi$'s such that

$$\mathrm{Lin}^{\mathbb{H}}_{\varphi}(X; Y)\ \subset\ \mathrm{Lin}^{\mathbb{C}}_{gh}(X; Y)\ .$$

The inclusion (5.4) says that the $\varphi$'s which satisfy

$$\varphi(g) = h \tag{5.7}$$

certainly work fine. The point is that they are the only $\varphi$'s which work, and Lemma 2.4 allows to compute them explicitly.
Assume for the moment that $X$ and $Y$ have the same chirality. By thinking of $\varphi = \vartheta_q$ for some quaternion $q$, Lemma 2.4 says that condition (5.7) is satisfied if and only if

$$q = \lambda(g+h) + \mu \left\{ 1 + \frac{2}{|g+h|^2}\, g \times h \right\} \qquad \text{with} \qquad \lambda, \mu \in \mathbb{R} \tag{5.8}$$

when $g + h \neq 0$, or

$$q \in \mathbb{R}^3 \qquad \text{such that} \qquad q \cdot g = 0 \tag{5.9}$$

when on the contrary $g + h = 0$. In the opposite chirality context one has to consider $\varphi = \overline{\vartheta}_q$ and the structural equation (5.7) becomes $\vartheta_q(g) = -h$: hence, the same result holds true, but for replacing everywhere $h$ with $-h$.
A general consequence must be retained from this discussion: whatever the chirality context and the complex structures are, the admissible $q$'s build a two dimensional real subspace of $\mathbb{H}$. This fact is the core of the next propositon.

**Proposition 5.7.** *Every complex linear map may be decomposed into the sum of two orthogonal quaternionic linear maps.*

*Proof.* Only the case where $X$ and $Y$ have the same chirality will be considered, the proof being similar in the opposite case. Denote by $\Lambda$ a complex linear map of type $(g, h)$, and distinguish two cases depending on the value of $g + h$. Consider first the case $g + h = 0$. Then (5.9) applies, and choosing $u$ and $v$ which complete $g$ to an orthonormal basis of $\mathbb{R}^3$, one has that $\Lambda = P + Q$, where the components $P$ and $Q$ are $\vartheta_u$ and $\vartheta_v$ quaternionic linear, respectively. Indeed, the identity $\Lambda(u * x) = P(u * x) + Q(u * x) = u * P(x) - u * Q(x)$ shows that

$$P(x) = \frac{\Lambda(x) - u * \Lambda(u * x)}{2} \qquad\qquad Q(x) = \frac{\Lambda(x) + u * \Lambda(u * x)}{2}$$

are the only possible candidates. It will be now proved that $P$ is quaternionic linear: the same arguments apply to $Q$. Since $P$ is real linear, it is sufficient to show that, for every $x$

$$P(u * x) = u * P(x) \qquad P(v * x) = -v * P(x) \qquad P(g * x) = -g * P(x)\ .$$

The first claim is true by construction, while the last one follows from the first two, inasmuch $g$ equals $uv$ up to the sign. The same argument also applies to the second claim, by taking into the account the complex linearity of $\Lambda$. Using indeed that $h = -g$, for every $x$

$$\Lambda(u * (v * x)) = -u * \{v * \Lambda(x)\}$$

due to the fact that $X$ and $Y$ have the same chirality, and similarly

$$\Lambda(v * x) = -\Lambda(v * (u * (u * x))) = v * \{u * \Lambda(u * x)\}\ .$$

Thus

$$\begin{aligned} 2\,P(v*x) &= \Lambda(v*x) - u*\Lambda((u*(v*x)) \\ &= v*\{u*\Lambda(u*x)\} - v*\Lambda(x) = -2\,v*P(x) \end{aligned}$$

for every $x$.

Similar, though lengthy, computations may be worked out also in the case $g+h \neq 0$. Hereafter an alternative and shorter proof is given. Again, two sub–cases should be distinguished, namely $g-h=0$ and $g-h \neq 0$: only the second, and more involved one, will be considered. Define

$$q_0 = g+h \qquad q_1 = 1 + \frac{2}{|g+h|^2}\, g\times h \qquad q_2 = g-h \qquad q_3 = 1 - \frac{2}{|g-h|^2}\, g\times h$$

and notice that they form a real orthogonal basis for $\mathbb{H}$. According to Proposition 4.1, one may write $\Lambda = \Lambda_0 + \Lambda_1 + \Lambda_2 + \Lambda_3$ where each $\Lambda_h$ is quaternionic linear with respect to $\omega_h = \vartheta_{q_h}$. Because of (5.8) one knows that

$$\Lambda_0 + \Lambda_1 \in \mathrm{Lin}^{\mathbb{C}}_{gh}(X;Y) \qquad\qquad \Lambda_2 + \Lambda_3 \in \mathrm{Lin}^{\mathbb{C}}_{g(-h)}(X;Y)\ .$$

Since $\Lambda \in \mathrm{Lin}^{\mathbb{C}}_{gh}(X;Y)$ by construction, the classical decomposition (5.2) yields $\Lambda_2 = \Lambda_3 = 0$. ∎

A conclusion must be retained form the proof of Proposition 5.7, which is relevant to regularity.

**Corollary 5.8.** *Assume that $\Lambda$ belongs to the class*

$$\mathrm{Lin}^{\mathbb{C}}_{g\,(-g)}(X;Y) \qquad \textit{or} \qquad \mathrm{Lin}^{\mathbb{C}}_{gg}(X;Y)$$

*according to whether $X$ and $Y$ have the same or different chiralities. Then $\Lambda$ is regular and moreover $s\{\Lambda\} \leq 2$.*

*Proof.* From the above quoted proof $\Lambda = P+Q$ where $P$ and $Q$ are imaginary quaternionic linear and quaternionic orthogonal, since $u$ and $v$ are real orthogonal imaginary units. ∎

The natural question is now if there are complex linear maps, other than the above quoted ones, which are regular. Next proposition answers in the positive, but says they are indeed quaternionic linear.

**Proposition 5.9.** *Assume that* $\Lambda \in \mathcal{R}(X;Y) \cap \mathrm{Lin}^{\mathbb{C}}_{gh}(X;Y)$ *and*

$$g+h \neq 0 \qquad \textit{or} \qquad g-h \neq 0$$

*according to whether $X$ and $Y$ have or not the same chirality. Then $s\{\Lambda\} \leq 1$.*

*Proof.* For definiteness, assume that $X$ and $Y$ have the same chirality, and refer to the proof of Proposition 5.7. One knows that $\Lambda = \Lambda_0 + \Lambda_1$, where $\Lambda_0$ is imaginary quaternionic linear because of $q_0 \in \mathbb{R}^3$, while $\Lambda_1$ is not unless $\Lambda_1 = 0$, due to $\mathrm{Re}\, q_1 \neq 0$ and Lemma 4.7. On the one hand, $\Lambda_0$ is regular due to the same lemma and hence $\Lambda_1 = \Lambda - \Lambda_0$ is regular too. If one now assumes that $\Lambda_1 \neq 0$, then $q_1 \in \mathbb{R}^3$ always due to Lemma 4.7: this fact contradicts $\mathrm{Re}\, q_1 \neq 0$. Thus $\Lambda_1 = 0$, which concludes the proof. ∎

All is now ready to complete the characterization of the regular maps of size two.

**Proposition 5.10.** $\Lambda \in \mathcal{R}(X;Y)$ *and* $s\{\Lambda\} \leq 2$ *if and only if there exists an imaginary unit* $g$*, such that* $\Lambda$ *belongs to*

$$\mathrm{Lin}^{\mathbb{C}}_{g(-g)}(X;Y) \qquad \textit{or} \qquad \mathrm{Lin}^{\mathbb{C}}_{gg}(X;Y) \tag{5.10}$$

*according to the whether* $X$ *and* $Y$ *have or have not the same chirality.*

Call *complex regularity* the complex linearity defined by (5.10) in the appropriate chirality context. It is worth to notice that it has exactly the same drawbacks of the regularity itself: for instance, the identity is never complex regular, nor is the composition of complex regular maps, in general. This fact, however, is not in contradiction with the usual rules of the complex calculus. For instance, the composition of complex linear maps is governed by (5.3) and is again complex linear: in particular, by composing complex regular maps one obtains again complex regular maps, but with respect to the wrong chirality context. Also the expectations about the identity are caused by a misunderstanding: to say that $X = Y$ holds at the level of sets, or even at the level of quaternionic spaces, does not imply that the same is true at the complex level. The point is that the complex regularity never concerns maps from a complex space into itself, so that there is no reason for the identity to play any special role.

*Proof.* The 'if part' is Corollary 5.8. Concerning the 'only if part', notice that $\Lambda$ is complex linear due to Proposition 5.6. Moreover, because of Proposition 5.9, the thesis is certainly true if $s\{\Lambda\} = 2$.
Assume now that $s\{\Lambda\} \leq 1$, namely that $\Lambda$ is quaternionic linear with respect to an imaginary $\varphi$. Due to (5.4), to conclude the proof it is enough to show that there exists an imaginary unit $g$ such that

$$\varphi(g) = -g \qquad \text{or} \qquad \varphi(g) = g$$

depending on the parity context. Assume that $X$ and $Y$ have the same chirality. Then $\varphi = \vartheta_q$ for some $q \in \mathbb{R}^3$ and, to have $\vartheta_q(g) = -g$, it is sufficient to take $g$ imaginary and real orthogonal to $q$. ∎

The previous proposition closes the problem of classifying a regular map $\Lambda$ by means of its quaternionic size. A different classification was proposed in [19], for the one–dimensional case, looking at the number $c\{\Lambda\}$ of the real independent imaginary units $g$, for which (5.10) holds. The two classifications are indeed equivalent, since it is not difficult to check that

$$c\{\Lambda\} = 3 - s\{\Lambda\} \ .$$

The only advantage of $s\{\Lambda\}$ is that it is based on a decomposition of $\mathrm{Lin}^{\mathbb{R}}(X;Y)$ and then, in principle, it is more directly computable.
The last part of the section, is devoted to the existing inclusions between the classes of the regular maps and that of the complex linear ones. On the one hand, a slight modification in the proof of Proposition 5.9 shows immediately that

$$\mathrm{Lin}^{\mathbb{C}}_{gh}(X;Y) \subset \mathcal{R}(X;Y)$$

if and only if $h = -g$ or $h = g$, depending on the chirality context. On the other hand, by a dimensional argument is is clear that regularity cannot be reduced to a single type of complex linearity. However, this is not so evident when the type of the complex linearity is allowed to vary, and one may wonder if

$$\mathcal{R}(X;Y) \subset \bigcup_{gh} \mathrm{Lin}^{\mathbb{C}}_{gh}(X;Y) \ .$$

The same question was raised in [19] for the case $X = Y = \mathbb{H}$, and answered in the negative by means of an interesting and powerful approach (see also [3] for a similar question). Hereafter, the same answer will be obtained by a very elementary dimensional argument.
The problem is to show that there exist regular maps of size three, which cannot be complex linear of any type. By definiteness, assume to work with left quaternionic spaces. A map $\Lambda \in \mathcal{R}(\mathbb{H})$ has $s\{\Lambda\} \leq 2$ if and only if

$$\big(\vartheta_g \,|\, \Lambda\big)_r = 0 \tag{5.11}$$

for some imaginary unit $g$. Fix now an orthonormal basis $e_1, e_2, e_3$ of $\mathbb{R}^3$, and write $\Lambda$ in components as

$$\Lambda(q) = \sum_{k=1}^{3} \vartheta_{e_k}(q)\ \alpha_k$$

where $\alpha_1, \alpha_2, \alpha_3$ vary in $\mathbb{H}$. By computing the quaternionic scalar product, via the rule expressed in Proposition 2.7, one obtains

$$\big(\vartheta_g \,|\, \Lambda\big)_r = -\sum_{k=1}^{3} (g \cdot e_k)\ g e_k\ \alpha_k = \sum_{k=1}^{3} (g \cdot e_k)\ \big[(g \cdot e_k) - g \times e_k\big]\ \alpha_k \ .$$

Consider for instance the case where $\alpha_1 = \alpha_2 = \alpha_3$, and denote by $\alpha$ their common value. Notice that

$$\sum_{k=1}^{3} (g \cdot e_k)\ g \times e_k = \sum_{h,k=1}^{3} (g \cdot e_h)(g \cdot e_k)\ e_h \times e_k = 0$$

and hence

$$\big(\vartheta_g \,|\, \Lambda\big)_r = \Big\{ \sum_{k=1}^{3} (g \cdot e_k)^2 \Big\}\ \alpha = \alpha \ .$$

Summing up, condition (5.11) is satisfied for no imaginary units $g$, as soon as $\alpha \neq 0$: thus, the corresponding regular $\Lambda$ has size exactly three.

## 6 Some nonlinear consequences

In this final section, a norm is added to the vector structures in order to make derivatives. Precisely, $X$ and $Y$ denote two quaternionic Banach spaces, $U$ an open subset of $X$, and $f : U \to Y$ is (one–time) differentiable in the real sense (i.e.

when $X$ and $Y$ are considered as real spaces). The regularity of $f$ in $U$ amounts to require that

$$df(x) \in \mathcal{R}(X;Y) \qquad \forall x \in U\ .$$

When $X$ and $Y$ are finite dimensional, it is well known that regularity implies real analyticity, and then smoothness, due to a Cauchy–type representation formula: see for instance [23] and [20]. The same is probably true also in the infinite dimensional case, by means of a reduction procedure similar to that of the complex case. However, this procedure is quite long and involved, and the result itself is not of interest here.

In the previous section it has been proved that, when $f$ is regular, then

$$s\{df(x)\} \leq 3$$

for every $x \in U$, due to the canonical decomposition of the differential itself. The aim here is to understand what happens when further restrictions are imposed on the quaternionic size of $df$. Apart from the trivial case of size zero, corresponding to the constant functions, the simplest nontrivial case is that of size one maps. The main result of this section states that, however, one obtains again trivial maps.

**Theorem 6.1.** *Assume that $X = Y = \mathbb{H}$ and $f$ is regular (and hence real analytic) in the open subset $U \subset \mathbb{H}$. If*

$$s\{df(x)\} \leq 1 \tag{6.1}$$

*for all $x \in U$, then the second derivative of $f$ vanishes identically on $U$.*

As a consequence, when $U$ is connected, the function $f$ becomes, up to translations, the restriction to $U$ of a single imaginary quaternionic linear map. Moreover, due to the real analyticity of $f$, the same conclusion holds under the weaker size restriction

$$\mathrm{int}\{x \in U : s\{df(x)\} \leq 1\} \neq \emptyset$$

where $\mathrm{int}(A)$ denotes the interior of a set $A$.

On the other hand, the conclusion of the theorem is false if the regularity assumption is removed. A remarkable counter–example is provided by the classical geometrical inversion

$$\zeta(x) = \frac{x}{|x|^2} = (\overline{x})^{-1} \tag{6.2}$$

on the set $U := \mathbb{H} \setminus \{0\}$. Indeed, it is not difficult to see that

$$d\zeta(x) \cdot v = -\zeta(x)\, \overline{v}\, \zeta(x) = -\overline{\vartheta}_{\zeta(x)}(v)\, \zeta(x)^2 \tag{6.3}$$

for every $x \neq 0$ and $v$. Hence, considering $\zeta$ as a map from a right space into a left one, its differential is quaternionic linear at each $x \in U$: the point is that $\overline{\vartheta}_{\zeta(x)}$ is imaginary if and only if $x$ is, so that the function $\zeta$ cannot be regular in any open subsets of $U$.

In fact, Liouville's Theorem on conformal mappings (see the Appendix B) guarantees that the inversions are essentially the only possible counter–examples. A real linear map $\Lambda : \mathbb{H} \to \mathbb{H}$ is conformal when, for some real scalar $\mu \geq 0$

$$|\Lambda(v)| = \mu\, |v| \qquad \forall v \in \mathbb{H} \tag{6.4}$$

and the conformality of the nonlinear map $f$ in $U$ means that the same property is satisfied by each $df(x)$ with $x \in U$, where $\mu$ possibly depends on $x$. The Liouville Theorem says that these maps are indeed the Möbius transformations. In [10] the result is shown to hold under very weak smoothness assumptions on $f$. In fact, smoothness is for free here, and simpler proofs are available in this case: for the reader convenience, the Nevanlinna proof [17] is sketched in the Appendix B, with the minor variants which the present context needs.
The function $\zeta$ defined by (6.2) is an example of conformal map, since $d\zeta(x)$ fulfills condition (6.4) with $\mu = |\zeta(x)|^2$: next lemma inserts this particular case into the appropriate general context.

**Lemma 6.2.** *Assume that $\Lambda \in \mathrm{Lin}^{\mathbb{R}}(\mathbb{H})$ is nontrivial. Then $\Lambda$ is quaternionic linear if and only if condition (6.4) is satisfied and*

$$\mathit{det}\ \Lambda \geq 0 \qquad \mathit{or} \qquad \mathit{det}\ \Lambda \leq 0$$

*according to whether the chiralities of the two involved copies of $\mathbb{H}$ are equal or different.*

*Proof.* It is based on the following two facts: all the automorphisms of $\mathbb{H}$ are positive rotations, while right and left multiplications by a quaternion $q$ are conformal maps of determinant $|q|^4$. Because of the representation (3.5), a quaternionic linear map $\mathbb{H} \to \mathbb{H}$ is then a conformal map. The sign of its determinant is decided by the type of the reference morphism: positive for automorphisms, which arises when the chiralities are the same, and negative in the opposite case.
Assume now that $\Lambda$ is a nontrivial conformal map. Because of (6.4), $\mu > 0$ and $\Lambda$ cannot vanish outside zero, so that one can introduce the map

$$\varphi(x) = \Lambda(x) * \Lambda(1)^{-1} .$$

As usual, the concrete action of $*$ depends on the chirality of that copy of $\mathbb{H}$, which acts as codomain of $\Lambda$. If $\diamond$ is the opposite operation in the same space, then $\Lambda = \Lambda(1) \diamond \varphi$ and the representation (3.5) shows that $\Lambda$ is quaternionic linear, as soon as $\varphi$ is either an automorphism or an anti–automorphism. The choice between the two options depends on the chirality context and, since $\det \varphi = \det \Lambda / |\Lambda(1)|^4$, it is decided by the determinant of $\Lambda$. By the very construction, $\varphi$ is a conformal map which is the identity on $\mathbb{R}$: thus it maps $\mathbb{R}^3$ into itself, and is a rotation there. The conclusion follows from the arguments in Section 2. ∎

*Proof of Theorem 6.1.* It is not restrictive to assume that $U$ is connected. Because of Lemma 6.2, the function $f$ is conformal in $U$. Then Liouville's Theorem (see Theorem B.1) applies to show that either the conclusion holds for $f$ itself, or it does for the function $z \mapsto f(x_0 + \zeta(z - x_0))$ on the domain $x_0 + \zeta(U - x_0)$, where $\zeta$ is the inversion defined by (6.2) and $x_0 \notin U$ is suitably chosen. The proof is complete if one shows that, in the second case, the function $f$ is constant on $U$. To this aim, write the real differential of $f$ in $U$ as

$$df(x) = \Lambda \circ d\zeta(x - x_0) \tag{6.5}$$

where $\Lambda$ is some suitable real linear conformal map, and then also quaternionic linear due to Lemma 6.2. The constancy of $f$ is equivalent to the triviality of $\Lambda$: in

fact, it follows from the triviality of $df(x)$ for some $x \in U$ (see also Lemma B.4). Assume by contradiction that $\Lambda \neq 0$, and compute the reference morphism of $df(x)$ from the previous formula. To this aim, it is convenient to fix a chirality context: from now on, $f$ will be assumed to act between right spaces, or briefly $f$ : right $\to$ right. The same arguments apply to all the other situations.
In this chirality context, the determinant of $df$ must be positive. Moreover, since the determinant of $d\zeta$ is negative, Lemma 6.2 says that one has necessarily to think of $\zeta$ : right $\to$ left and $\Lambda$ : left $\to$ right. Denote by $\overline{\vartheta}_q$ the reference morphism of $\Lambda$, for some $q \neq 0$. From (6.3), it follows that the reference morphism of $df(x)$ is

$$\overline{\vartheta}_q \circ \overline{\vartheta}_{\zeta(x-x_0)} = \vartheta_{q\,\zeta(x-x_0)}$$

for every $x \in U$. Because of Lemma 4.7, to reach a contradiction with the regularity of $f$ in $U$, it is sufficient to show that the condition

$$q\,\zeta(x - x_0) \in \mathbb{R}^3$$

cannot be identically satisfied in $U$. But $q\,\zeta(x - x_0)$ is imaginary if and only if $q \cdot \overline{x - x_0} = 0$, which can be only satisfied in a meager subset of $U$. ∎

Besides its use in the previous proof, Lemma 6.2 also provides a very convenient answer to the question raised in [13], namely: how to describe the conformality of a function $f : U \subset \mathbb{H} \to \mathbb{H}$ as a differentibility condition, expressed in term of differential quotients. The small price one has to pay, is to choose the chirality context according to the sign of the determinant of $f$. Notice that, if $U$ is connected, then this sign cannot change in $U$ (see Lemma B.4). Choose, for definiteness, the case of positive determinants and left chiralities for both the copies of $\mathbb{H}$. Then, as a straightforward consequence of Lemma 6.2, the map $f$ is conformal in $U$ if and only if the following condition is satisfied: *for every $x \in U$, there exists an $\omega_x \in \mathrm{Aut}(\mathbb{H})$ such that*

$$\lim_{v \to 0} \left[\omega_x(v)\right]^{-1} \left[f(x + v) - f(x)\right] \tag{6.6}$$

*does exist in* $\mathbb{H}$. In [13], the construction of the quotient is more involved, inasmuch the authors seem to ignore they are dealing with automorphisms of $\mathbb{H}$.

*Remark* 6.3. The characterization provided in [13] only depends on the chirality of the copy of $\mathbb{H}$ acting as codomain: moreover, for every choice of this chirality, all the conformal maps may be represented, independently of the sign of the determinant of their differentials. The ultimate reason of that, is the use in [13] of *structural sets*, instead of standard bases: they are always made of orthonormal bases of $\mathbb{R}^3$ (depending on the point), but their orientations are now arbitrary. Of course, the result in [13] is not in contradiction with the one stated above: indeed, once the chirality of codomain is given, that of the domain may be tuned on the function to be represented.

In the differential quotient (6.6), the automorphism may depend on the point: for instance, this is exactly what happens for the inversions (6.3), though in a different chirality context. Next lemma says that, in order to avoid trivialities, $\omega$ is forced to vary with $x$.

**Proposition 6.4.** *Assume that the function $f$ is one–time real differentiable in $U$ and that, for some $\varphi$, it satisfies*

$$df(x) \in \mathrm{Lin}^{\mathbb{H}}_{\varphi}(X;Y) \qquad \forall x \in U\,. \tag{6.7}$$

*Then $f$ is smooth and its second derivative vanishes identically on $U$.*

The statement is a minor extension of a well known fact, at least when $X = Y = \mathbb{H}$ and $\varphi$ is the identity function. Its history is well described in [5] and [25]. In the last paper, the general finite–dimensional case is also treated, while the most cited one–dimensional proof is that of [23]. With respect to these results, the main difference of Theorem 6.4 stays in the proof, which makes a direct and trivial use of the non commutativity of $\mathbb{H}$: because of that, it is suitable for further extensions, like to maps in Clifford algebras (see [18]).

*Proof.* Fix an arbitrary imaginary unit $g$ and set $h = \varphi(g)$. Then endow $X$ and $Y$ with the complex structures associate to $g$ and $h$ respectively, and notice that

$$df(x) \in \mathrm{Lin}^{\mathbb{C}}_{gh}(X;Y) \qquad \forall x \in U\,.$$

In other words $f$ is complex differentiable, or holomorphic, in $U$. This is equivalent to complex analyticity and implies real analyticity and then smoothness. All that is widely known in the one–dimensional complex case, but it is a standard fact also in the general case: for instance, see [16]. Hereafter just two real derivatives are needed.
Since $\mathrm{Lin}^{\mathbb{H}}_{\varphi}(X;Y)$ is a closed subspace of $\mathrm{Lin}^{\mathbb{R}}(X;Y)$, with respect to the standard operatorial norm, one has

$$d\big(df(x)\big)\cdot v \in \mathrm{Lin}^{\mathbb{H}}_{\varphi}(X;Y)$$

for every $x \in U$ and every $v \in X$. In other words, for every $x \in U$, the second derivative

$$d^2 f(x)\cdot(u,v) := d\big(df(x)\cdot u\big)\cdot v$$

is quaternionic linear, with respect to $\varphi$, in the second variable: in fact, the symmetry of the second derivative says that the same is also true for the first variable. By using the two partial homogeneities, in a given order and in the reversed one, one gets the equality

$$\varphi(p\,q) * d^2 f(x)\cdot(u,v) \;=\; d^2 f(x)\cdot(p*u, q*v) \;=\; \varphi(qp) * d^2 f(x)\cdot(u,v)$$

for every choice of $x \in U$, $u,v \in X$ and $p,q \in \mathbb{H}$. Since $\varphi$ is bijective and the product in $\mathbb{H}$ is non commutative, $d^2 f(x)\cdot(u,v) = 0$ for every choice of the variables. ∎

A comment is maybe worth, about the complex holomorphy casted into the quaternionic framework, used in the proof. The reader might believe that, due to the non commutativity of $\mathbb{H}$, only few functions $f$ may really satisfy

$$df(x) \in \mathrm{Lin}^{\mathbb{C}}_{gh}(X;Y) \qquad \forall x \in U \tag{6.8}$$

unless $g$ and $h$ are suitably chosen, and probably tuned on the chirality context. On the contrary, for every choice of the (unrelated) imaginary units $g$ and $h$ and every choice of the chirality context, condition (6.8) selects a quite large class of functions: roughly speaking, as large as one may expect in the standard, commutative framework.
To be convinced of that, consider once more the case where $X = Y = \mathbb{H}$ is a left space. Then identify with $\mathbb{C}^2$ the copy of $\mathbb{H}$ which plays the role of $X$, by means of

$$\Phi(z, w) \;=\; \mathrm{Re}\,(z) + \mathrm{Im}\,(z)\; g + \left\{\mathrm{Re}\,(w) + \mathrm{Im}\,(w)\; g\right\} u$$

where $u$ is an arbitrary imaginary unit, which is moreover orthogonal to $g$. The first component $z \mapsto$ Re $(z)$ + Im $(z)$ $g$ is the real algebra insertion $\mathbb{C} \to \mathbb{H}$, which defines the complex structure of $\mathbb{H}$: by construction, it is a complex linear map. Because of the correct position of $u$ (see Section 3), the same happens to the second component of $\Phi$. Summing up, $\Phi$ is a complex linear isomorphism $\mathbb{C}^2 \cong \mathbb{H}$. Do the same with $h$ for the other copy of $\mathbb{H}$, that playing the role of $Y$: denote by $\Psi$ the complex linear isomorphism obtained in this way. It is clear that

$$L \in \mathrm{Lin}^{\mathbb{C}}(\mathbb{C}^2;\mathbb{C}^2) \;\mapsto\; \Psi \circ L \circ \Phi^{-1} \in \mathrm{Lin}^{\mathbb{C}}_{gh}(\mathbb{H})$$

is a linear isomorphism, at least of real spaces (for complex linearity, see Remark 5.2). The identification also works at a nonlinear level, since $\Phi$ and $\Psi$ do not change with the point. The abundance of holomorphic maps $f : U \subset \mathbb{H} \to \mathbb{H}$ then depends on the following characterization: they are exactly the maps which factorize into

$$f = \Psi \circ F \circ \Phi^{-1}$$

where $F : \Phi^{-1}(U) \subset \mathbb{C}^2 \to \mathbb{C}^2$ is holomorphic in the standard sense.

Coming back to regular functions, the previous arguments show that the size restriction

$$s\{df(x)\} \leq 2 \qquad \forall x \in U \tag{6.9}$$

has not the same drawbacks of (6.1). Indeed, due to Proposition 5.10, condition (6.9) is equivalent to

$$df(x) \in \mathrm{Lin}^{\mathbb{C}}_{g_x h_x}(X;Y) \qquad\qquad \forall x \in U$$

for some suitable choice of the imaginary unit $g_x$ and

$$h_x = -g_x \qquad\qquad h_x = g_x$$

depending on whether $X$ and $Y$ have or not the same chirality. When $g_x = g$ is independent of $x$, this becomes a standard complex holomorphy assumption: just above it has been shown that there are many of these functions. The situation changes when $g_x$ varies with $x$. In the literature, these functions are usually called *almost holomorphic*, with respect to the almost complex structures defined by $g_x$ on the tangent space $T_x U = X$, and $h_x$ on $T_{f(x)} Y = Y$. The choice of $g_x$ strongly affects the class of the almost holomorphic functions, in a way which seems not so easy to describe: this discussion is not within the scope of the present paper.

## A Quaternionic scalar products

Let $X$ be a quaternionic vector space and $*$ its scalar multiplication. There are no a priori restrictions on the chirality of $X$. A quaternionic scalar product on $X$ is essentially a word–by–word translation of a complex sesquilinear form, into a quaternionic framework. To make clear the consequences of the non commutativity of $\mathbb{H}$, the axioms will be divided into two groups. The first one contains all the axioms which do not depend on the scalar multiplication, namely the request that

$$\begin{aligned} &(x+y\,|\,z) = (x\,|\,z) + (y\,|\,z) \\ &(y\,|\,x) = \overline{(x\,|\,y)} \\ &(x\,|\,x) > 0 \quad \text{if} \quad x \neq 0 \end{aligned}$$

for all $x, y, z \in X$. The second group is made by a single prescription, which concerns homogeneity. In the complex framework one has to decide whether the homogeneity is referred to the left member of the scalar product, or to the right one: the second choice may be also seen as a conjugate homogeneity with respect to the left member. Due the non commutativity of $\mathbb{H}$, the two choices of the complex case seem to split into the following four different choices

$$(q * x\,|\,y) \;=\; \begin{cases} q(x\,|\,y) \\ (x\,|\,y)\overline{q} \end{cases} \tag{A.1}$$

and

$$(q * x\,|\,y) \;=\; \begin{cases} (x\,|\,y)q \\ \overline{q}(x\,|\,y) \end{cases} \tag{A.2}$$

for all $x, y \in X$ and all $q \in \mathbb{H}$. Notice that, by conjugating a quaternionic scalar product which satisfies one of the two axioms in (A.1), one gets a quaternionic scalar product satisfying the other axiom in (A.1). Of course, the same happens for (A.2).

The point is that, due the non commutativity of $\mathbb{H}$, not all the choices are equally possible. Not surprisingly, the restriction is driven by the chirality of $X$: it is not difficult to see that the axioms in (A.1) are suitable for left quaternionic spaces only, while those in (A.2) are for right quaternionic spaces.

To be concrete, imagine that $X$ is a left quaternionic space and that the first axiom in (A.2) is satisfied. Then one should have

$$(x\,|\,y)(p\,q) = ((p\,q) * x\,|\,y) = (p * (q * x)\,|\,y) = (q * x\,|\,y)p = (x\,|\,y)(q\,p)$$

for every $x, y \in X$ and $p, q \in \mathbb{H}$. This would imply that $(x\,|\,y) = 0$ for every $x, y \in X$, which of course is possible only in the trivial and uninteresting case $X = \{0\}$. Similar conclusions hold in all the forbidden cases.

Before concluding, a couple of remarks are maybe worth. The first concerns further generalizations of the homogeneity property of the quaternionic scalar product, involving automorphisms of $\mathbb{H}$ in the spirit of (3.2): though this is certainly possible, it is not of interest here.

The second one concerns the special case $X = \mathbb{H}$. It is not difficult to see that, if

$\mathbb{H}$ is seen as a left space over itself, the only possible quaternionic scalar products are given by the conjugate pair

$$(p\,|\,q) \;=\; p\,\overline{q} \qquad\qquad (p\,|\,q) \;=\; q\,\overline{p}\,.$$

Of course, this is true up to a multiplicative and positive real factor, representing the value of $(1\,|\,1)$. Analogously, the conjugate pair

$$(p\,|\,q) \;=\; \overline{q}p \qquad\qquad (p\,|\,q) \;=\; \overline{p}\,q$$

is obtained when $\mathbb{H}$ is a right quaternionic space.

## B The Liouville theorem on conformal maps

Let $U \subset \mathbb{R}^n$ a connected open set and $f : U \to \mathbb{R}^n$ a smooth map which, at any $x \in U$, satisfies the following differential assumption: there is number $\mu(x) \geq 0$ such that

$$|df(x)\cdot v| \;=\; \mu(x)\;|v| \qquad\qquad \forall v \in \mathbb{R}^n\,. \tag{B.1}$$

Such a map $f$ is called conformal in $U$, as far as the linear map $df(x)$ for every $x \in U$: the coefficient $\mu(x)$ is said to be the similarity ratio of $df(x)$. When $n = 1$ or $n = 2$, these maps are quite abundant: this is no longer true in higher dimensions because of the following result.

**Theorem B.1.** *Assume that $n \geq 3$. Then either the second derivative of $f$ vanishes identically on $U$, or there is an $x_0 \in \mathbb{R}^n \setminus U$ such that this is true for the map $z \mapsto f(x_0 + \zeta(z - x_0))$ on the set $x_0 + \zeta(U - x_0)$.*

The second case cannot occur when $U = \mathbb{R}^n$. Here $\zeta$ denotes the inversion in $\mathbb{R}^n$ with center in the origin, defined by

$$\zeta(z) \;=\; \frac{z}{|z|^2}\,.$$

for every $z \neq 0$, as in (6.2). It is not difficult to check that $\zeta$ is an involutive diffeomorphism of $\mathbb{R}^n \setminus \{0\}$, and that the same is true for the map $z \mapsto x_0 + \zeta(z - x_0)$ with respect to $\mathbb{R}^n \setminus \{x_0\}$. The statement is the same of the classical Liouville Theorem, but for the lack of invertibility assumptions on $f$ and its first derivative: avoiding them is relevant for Theorem 6.1. In this appendix, a standard proof in the literature will be sketched, by pointing out the minor modifications which are needed to overcame the invertibility problems (see also [10]): the reference proof is the expanded version of the original Nevanlinna proof, which is contained in the classical Berger textbook [1] (pages 222–226).
Start by considering the map $\mu : U \to Y$ of the similarity ratios, defined in (B.1). Fixing any $v$ with $|v| = 1$, makes clear that $\mu$ is continuous in $U$, and also smooth in the open set

$$V = \{x \in U : \mu(x) > 0\}\,.$$

To rule out the trivial case of a constant $f$, assume that $V \neq \emptyset$. Notice that, at all the points of $V$, the injective linear map $df(x)$ is in fact an isomorphism.
Next lemma says how the inverse of the similarity ratio

$$\rho(x) = \frac{1}{\mu(x)}$$

behaves on the set $V$: this is the true core of the Liouville Theorem.

**Lemma B.2.** *Assume that $n \geq 3$. Then $u \cdot v = 0$ implies $d^2\rho(x) \cdot (u,v) = 0$ for every $x \in V$.*

The proof is covered by the paragraphs 9.5.4.9–15 in [1]. Essentially, it is obtained it by differentiating the expression

$$(df(x) \cdot u) \cdot (df(x) \cdot v) = 0$$

(the central $\cdot$ is the Euclidean real scalar product in $\mathbb{R}^n$) and by manipulating the result, until reaching the equality

$$\begin{aligned}(d\rho(x) \cdot v)\, df(x) \cdot u \;+\; (d\rho(x) \cdot u)\, df(x) \cdot v \,+ \\ \rho(x)\, d^2 f(x) \cdot (u,v) \;=\; 0\,.\end{aligned} \tag{B.2}$$

Then one more derivative is taken and, to complete the proof, one has to look at the result at the light of a basic symmetry principle: in [1], this is named the Braid lemma.
Standard results now apply to show that, for every $x \in V$, $d^2\rho(x)$ must be proportional to the Euclidean scalar product, namely

$$d^2\rho(x) \cdot (u,v) \;=\; \sigma(x)\; \{u \cdot v\} \qquad\qquad \forall u, v \in \mathbb{R}^n$$

for some suitable real $\sigma(x)$. The smoothness of $\rho$ yields that of $\sigma$, and another well interpreted session of derivatives yields

$$d\sigma = 0 \qquad \text{on} \qquad V$$

(see 9.5.4.16 in [1]). Thus $\sigma$ is constant on each connected component of $V$: let $W$ be one of them, which is open inasmuch $V$ is.
In all that, the invertibility assumptions did not play any role: they enter now to predict the analytical expression of $\rho$ in the set $W$. Indeed, the equation

$$d^2\rho(x) \cdot (u,v) \;=\; \sigma\; \{u \cdot v\}$$

can be integrated by inspection, yielding the following result.

**Lemma B.3.** *There exist a vector $x_0 \in \mathbb{R}^n$ and two real constants $A$ and $B$ such that*

$$\rho(x) = A|x - x_0|^2 + B \qquad\quad \forall x \in W\,.$$

*Moreover, either $A = 0$ or $B = 0$.*

Some remarks are worth. By the very construction of $\rho$, it cannot occur that $A = B = 0$ and moreover $x_0 \notin U$ when $B = 0$. In particular, if $U = \mathbb{R}^n$ then necessarily $A = 0$.
The proof of Lemma B.3 is covered by the paragraphs 9.5.4.17–20 in [1], under the assumption that $f$ is globally invertible in $W$. Looking at it, however, one immediately realizes that only the *local* invertibility of $f$ in $W$ is used: this is automatically granted at all points of $V$ by the Inverse Mapping Theorem, with no need of ad hoc assumptions.
The integration result applies to all the connected components of $V$, and the last step is to export it to the all set $U$. In principle, one should worry about the possible lack of local invertibility at the points of $V \setminus U$: next lemma says that this is an artificial question.

**Lemma B.4.** *It results* $V = U$.

*Proof.* Let $W$ be a connected component of $V$, which is nonempty by hypothesis: the proof is complete if one shows that $W = U$. Assume by contradiction that $W \neq U$. Since $W$ is open, $W \cup \partial_U W \cup \mathrm{int}(U \setminus W)$ is a partition of $U$: the connection of $U$ then yields $\partial_U W \neq \emptyset$. Choose now $x_* \in \partial_U W$ and notice that much more than $x_* \notin W$ is true: indeed, necessarily $x_* \notin V$. Otherwise $B(x_*, r) \subset V$ for some $r > 0$ and hence $W \cup B(x_*, r) \subset V$ would be a connected, strict superset of $W$: this would contradict the fact that $W$ is a connected component.
Summing up, one knows that $\mu(x_*) = 0$. On the other hand, Lemma B.3 guarantees that

$$\mu(x) = \frac{1}{A|x - x_0|^2 + B} \qquad \forall x \in W$$

for some suitable choice of the involved parameters. Thus, from the continuity of $\mu$ in $U$ one deduces that

$$\mu(x_*) = \lim_{W \ni x \to x_*} \mu(x) = \frac{1}{A|x_* - x_0|^2 + B} \neq 0$$

which contradicts the previous conclusion. ■

All is now ready to conclude the proof of the Liouville Theorem, using Lemma B.3 with $W = V = U$ and distinguishing two different cases.
Assume first that $A = 0$. In this case, $\rho(x)$ does not depend on $x$ and from equation (B.2) one deduces that

$$d^2 f(x) \cdot (u, v) = 0$$

for every $x \in U$ and every orthogonal pair $u, v \in \mathbb{R}^n$. In fact, the same is a posteriori true for every pair of vectors $u$ and $v$, and hence $f$ is the restriction to $U$ of an affine linear map on $\mathbb{R}^n$.
Assume now that $B = 0$, and hence also that $x_0 \in \mathbb{R}^n \setminus U$. The composed map $g(z) = f(x_0 + \zeta(z - x_0))$ is again conformal and, by computing, one finds

$$\begin{aligned} |dg(z) \cdot v| &= |df(x_0 + \zeta(z - x_0)) \cdot d\zeta(z - x_0) \cdot v| \\ &= \frac{1}{A|\zeta(z - x_0)|^2} \frac{1}{|z - x_0|^2} |v| \;=\; \frac{1}{A} |v| \end{aligned}$$

for every $z$ in the open connected set $x_0 + \zeta(U - x_0)$ and every $v \in \mathbb{R}^n$. Hence, the same arguments used above allow to conclude that $g$ is affine.

## References


[1] M. BERGER. *Geometry I.* Universitext, Springer-Verlag, Berlin, 1994.

[2] F. BRACKX, R. DELANGHE AND F. SOMMEN. *Clifford Analysis.* Research Notes in Mathematics 76, Pitman, Boston, 1982.

[3] J. CHEN AND J. LI Quaternionic maps between hyperkhler manifolds. *J. Differential Geom.* **55 (2)** (2000), 355–384.

[4] C.G. CULLEN. An integral theorem for analytic intrinsic functions on quaternions. *Duke Math. J.* **32** (1965), 139–148

[5] R. DELANGHE, R.S KRAUSSHAR AND H.R.MALONEK. Differentiability of functions with values in some real associative algebras: approaches to an old problem. *Bull. Soc. Roy. Sci. Liège.* **70 (4-6)** (2001), 231–249.

[6] R. FUETER Die Funktionentheorie der Differentialgleichungen $\Delta u = 0$ und $\Delta\Delta u = 0$ mit vier reellen Variablen. *Comment. Math. Helv.* **7 (1)** (1935), 307–330.

[7] K. GÜRLEBECK, K. HABETHA AND W. SPRÖSSIG *Holomorphic functions in the plane and n-dimensional space.* Birkhuser Verlag, Basel, 2008.

[8] G. GENTILI, C. MARICONDA AND M. TARALLO. Quaternionic regular maps and $\overline{\partial}$-type operators. *Simon Stevin.* **67 (3–4)** (1993), 333–359.

[9] G. GENTILI AND D. STRUPPA. A new theory of regular functions of a quaternionic variable. *Adv. Math.* **216** (2007), 279–301.

[10] T. IWANIEC AND G. MARTIN. The Liouville theorem. *Analysis and topology*, 339361, World Sci. Publ., New Jork, 1998.

[11] H. LEUTWILER. Modified quaternionic analysis in $R^3$. *Complex Variables Theory Appl.* **20 (1–4)** (1992), 19–51.

[12] G. LAVILLE AND I. RAMADANOFF. Holomorphic Cliffordian functions. *Adv. Appl. Clifford Algebras.* **8 (2)** (1998), 323–340.

[13] R.S. KRAUSSHAR AND H.R. MALONEK. A characterization of conformal mappings in $\mathbb{R}^4$ by a formal differentiability condition. *Bull. Soc. Roy. Sci. Liège.* **70 (1)** (2001), 35-49.

[14] V. KRAVCHENKO, M.V. SHAPIRO. *Integral representations for spatial models of mathematical physics.* Pitman Research Notes in Mathematics Series (351), Longman, Harlow, 1996.

[15] H. MALONEK. A new hypercomplex structure of the Euclidean space $R^{m+1}$ and the concept of hypercomplex differentiability. *Complex Variables Theory Appl.* **14 (1–4)** (1990),25–33.

[16] J. MUJICA. Complex analysis in Banach spaces. Holomorphic functions and domains of holomorphy in finite and infinite dimensions. *North-Holland Mathematics Studies, 120. Notas de Matemtica (107).*, North-Holland Publishing Co., Amsterdam, 1986

[17] R. NEVANLINNA On differentiable mappings. *Analytic functions*, 3–9, Princeton Univ. Press, Princeton–N.J., 1960.

[18] K. NÔNO. Characterization of Clifford differentiable functions. *Finite or infinite dimensional complex analysis (Fukuoka, 1999)*, 395–400, Lecture Notes in Pure and Appl. Math., 214, Dekker, New York, 2000.

[19] A. PEROTTI Holomorphic functions and regular quaternionic functions on the hyperkhler space $\mathbb{H}$. *More progresses in analysis: proceedings of the 5th international Isaac congress*, World Scientific, 2008.

[20] D. PERTICI Regular functions of several quaternionic variables. *Ann. Mat. Pura Appl.* **151 (4)** (1988), 39-65.

[21] A.J. SOMMESE. Quaternionic manifolds. *Math. Ann.* **212** (1974/75), 191–214.

[22] V. SOUCEK. Holomorphicity in quaternionic analysis. *Geometry seminars, 1982–1983 (Bologna, 1982/1983)*, 147–153, Univ. Stud. Bologna, Bologna, 1984.

[23] A. SUDBERY. Quaternionic analysis. *Math. Proc. Cambridge Philos. Soc.* **85 (2)** (1979), 199–224.

[24] P.B. YALE. Automorphisms of the Complex Numbers. *Mathematics Magazine.* **39 (3)** (1966), 135–141.

[25] O. YASUKURA. On Ehresmann's theorem of quaternion holomorphy. *Yokohama Math. J.* **49 (2)** (2002), 163–169.

[26] F. ZHANG. Quaternions and matrices of quaternionis. *Linear Algebra and its applications.* **251** (1997), 21–57.

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