Certain Conformal-like Infinitesimal Symmetries and the Curvature of a Compact Riemannian Manifold

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Abstract

The notion of orthogonally conformal vector field on a Riemannian manifold is introduced. This class of vector fields properly includes the normalization of nowhere zero conformal ones. It is clarified in several examples. An integral inequality which relates the existence of orthogonally conformal vector fields with properties of the Ricci tensor of a compact Riemannian manifold is proved and some applications are shown.

1 Introduction

Relating the existence of geometrically relevant vector fields on a Riemannian manifold with its curvature properties is a classical topic in Differential Geometry. Recall for instance the well-known Bochner's technique ([7] and references therein). In this note, we introduce a new class of geometrically relevant vector fields on Riemannian manifolds and obtain an integral inequality under the assumption of the existence of such vector fields. The new notion has been inspired by the behavior of the unit vector field tangent to the parallels of a surface of revolution in the 3-dimensional Euclidean space. A flow of such a vector field provides a linear isometry on its orthogonal complement at any point but, in

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general, this vector field fails to be Killing unless the surface is a cylinder (see details in Example 1). This situation can be considered as a particular case of a Riemannian submersion with 1-dimensional fibers. In this case, each (local) flow of the unit vertical vector field gives linear isometries on horizontal spaces of the submersion. But the unit vertical vector field of a such Riemannian submersion is Killing if and only if the fibers are totally geodesics (Example 7). On the other hand, if we consider a conformal (resp. Killing) vector field X on a Riemannian manifold (M, g), i.e. the Lie derivative with respect to X of g satisfies $\mathcal{L}_X g = 2\rho g$ (resp. $\mathcal{L}_X g = 0$), and we assume that X has no zeroes, then $Z = (1/\sqrt{g(X,X)})\, X$ satisfies $(\mathcal{L}_Z g)(U, V) = 2\rho g(U, V)$ (resp. $(\mathcal{L}_Z g)(U, V) = 0$), for all $U, V \perp Z$. Therefore, any of the flows Φ_t of the vector field Z induces a linear conformal (resp. isometry) mapping from $Z_p^\perp$ onto $Z_{\Phi_t(p)}^\perp$.

Thus, it seems natural to introduce the following notion: an *orthogonally conformal* vector field is a unit vector field Z on a Riemannian manifold, (M, g), which satisfies

$$(\mathcal{L}_Z g)(U, V) = 2\rho g(U, V), \tag{1}$$

for all $U, V \perp Z$. In particular, when $\rho = 0$, we will call Z *orthogonally Killing*. This class of vector fields properly includes the normalized vector fields of the nowhere zero conformal vector fields (see Section 2). That is, there exist unit vector fields which satisfy (1) but they cannot be obtained from a nowhere zero conformal vector field. Even more, there exists a family of compact Riemannian manifolds such that each one of its members admits a unit vector field satisfying (1) although no conformal vector field is free of zeroes (Example 6).

The main aim of this note is to relate the existence of such orthogonal conformal vector fields on a compact Riemannian manifold with its curvature properties. So, we will obtain a vanishing result in the spirit of the well-known Bochner's technique as follows.

Theorem 1. *Let (M, g) be an $n(\geq 3)$-dimensional compact Riemannian manifold. If (M, g) admits an orthogonally conformal vector field Z, then*

$$\int_M \mathrm{Ric}(Z, Z)\, d\mu_g \geq 0,$$

where Ric *denotes the Ricci tensor of (M, g) and $d\mu_g$ is the canonical measure induced by the Riemannian metric g. Moreover, the equality holds if and only if $\nabla_U Z = 0$ for any $U \perp Z$, and in such case, Z is orthogonally Killing.*

As an application, we obtain two consequences.

Corollary 1. *If a compact Einstein Riemannian manifold (M, g), with* $\dim M \geq 3$ *and* $\mathrm{Ric} = \lambda g$*, admits an orthogonally conformal vector field, then $\lambda \geq 0$.*

The second one can be regarded as a Wu-type result [7] for orthogonally conformal vector fields.

Corollary 2. *If the Ricci tensor of a $n(\geq 3)$-dimensional compact Riemannian manifold, (M, g), is negative semi-definite everywhere and negative definite at some $p \in M$, then (M, g) admits no orthogonally conformal vector field.*

2 Examples

It should be recalled that the existence of a nowhere zero vector field on a manifold M imposes some restrictions on its topology. In fact, it holds if and only if M is noncompact or its Euler-Poincaré number vanishes. If $\dim M = 2$, then every unit vector field is orthogonally conformal. Therefore, when $\dim M = 2$, (M, g) admits an orthogonally conformal vector field if, and only if, either M is noncompact or M is compact and its Euler-Poincaré characteristic vanishes. We would like to point out that if Z is an orthogonally conformal vector field for a Riemannian metric g, then the vector field $\left(1/\sqrt{g^1(Z,Z)}\right) Z$ is also orthogonally conformal for every Riemannian metric g^1 conformally related to g.

Example 1. Let $x(v)$ and $z(v)$ be smooth functions on an open interval $]a, b[$ with $x(v) > 0$ for every $v \in]a, b[$. Let S be the surface of revolution in the Euclidean space $\mathbf{E}^3$ with rotation axis z obtained from the above data. Thus, a parametrization for S is given by

$$\mu(u, v) = \big(x(v)\cos u, x(v)\sin u, z(v)\big).$$

We consider the unit vector field tangent to parallels, $Z \in \mathfrak{X}(S)$, given by

$$Z(\mu(u,v)) = \frac{1}{x(v)}\frac{\partial \mu}{\partial u}\,|_{(u,v)}\,.$$

From the well-known formulas for the Christoffel symbols of a surface of revolution, we deduce that Z is orthogonally Killing. If we suppose that Z is Killing then Z is a geodesic vector field. Therefore all the parallels of the surface of revolution S are geodesics and so x is constant, and S is a cylinder.

Example 2. Let $(\mathbb{T}^2, g)$ be a 2-dimensional flat Riemannian torus and let Z be a unit vector field on $\mathbb{T}^2$ which is not parallel. We assert that the orthogonally conformal vector field Z is not the normalization of a conformal vector field K. Indeed, if such K exists, then as a consequence of the classical result by H. Wu [7], which improved a previous one by K. Yano [8], the conformal vector field K is parallel. Hence Z must be also parallel. This contradicts our choice of Z.

Example 3. Let (I, dt^2) be an open interval endowed with its standard metric and let (M, g) be a Riemannian manifold. Given a smooth function $f > 0$ defined on $\widetilde{M} = I \times M$, we consider on $\widetilde{M}$ the twisted metric $\tilde{g} = \pi_I^* dt^2 + f^2 \pi_M^* g$, where π_I and π_M are the corresponding projections onto I and M, respectively. The vector field ∂_t on $\widetilde{M}$ is orthogonally conformal. Indeed, given $U, V \perp \partial_t$, clearly $[\partial_t, U] = [\partial_t, V] = 0$ and a direct computation shows that the Lie derivative of $\tilde{g}$ with respect to ∂_t satisfies

$$(\mathcal{L}_{\partial_t}\tilde{g})(U, V) = 2\,\partial_t(\log(f))\,\tilde{g}(U, V).$$

This computation also holds if (I, dt^2) is replaced by the 1-sphere $(\mathbb{S}^1, d\theta^2)$. Therefore, for M compact, we obtain orthogonally conformal vector fields on compact Riemannian manifolds.

Remark 1. In Example 3, if the function f only depends on t then $\tilde{g}$ is in fact the warped product of dt^2 and g with warping function f. In this case, it is not difficult to show that $\nabla_X(f\partial_t) = f'X$ for any vector field X on $\widetilde{M}$, [3, Prop. 7.35]. Therefore, $f\partial_t$ is conformal with $\mathcal{L}_{f\partial_t}\tilde{g} = 2f'\tilde{g}$ and ∂_t is the normalized vector field obtained from $f\partial_t$. Clearly, the function $\partial_t(\log(f))$ is now constant along each slice $\{t\} \times M$.

Example 4. In order to show examples of orthogonally conformal vector fields which cannot be obtained from nowhere zero conformal vector fields, we add in Example 3 the condition that the twisting function f satisfies that $\partial_t(\log(f))$ is not constant along each slice $\{t\} \times M$. Suppose now that ∂_t is the normalization of a nowhere zero conformal vector field, i.e. for a certain smooth function $h > 0$ defined on $\widetilde{M}$, the vector field $h\,\partial_t$ is conformal. Given $U \perp \partial_t$, direct computations show

$$U(h) = 0 \qquad \text{and} \qquad \partial_t(\log(h)) = \partial_t(\log(f)).$$

Therefore,

$$U(\partial_t(\log(f))) = U(\partial_t(\log(h))) = \partial_t(U(\log(h))) = 0,$$

which implies that $\partial_t(\log(f))$ only depends on t, contrary to our assumption.

Example 5. Let us consider the usual Hopf fibration $\pi : \mathbb{S}^{2n+1} \longrightarrow \mathbb{C}P^n$ from the odd dimensional unit sphere $\mathbb{S}^{2n+1} = \{(z_1, ..., z_{n+1}) \in \mathbb{C}^{n+1}\colon \sum_{j=1}^{n+1} \mid z_j \mid^2 = 1\}$ onto the complex projective space $\mathbb{C}P^n$ endowed with its Fubini-Study metric g_{FS} of constant holomorphic sectional curvature 4. For each $u \in C^\infty(\mathbb{S}^{2n+1})$ we set $g_u = e^u\,\pi^* g_{FS} + \omega \otimes \omega$, where ω is the 1-form naturally obtained from the usual connection on this fibre bundle. Each g_u is a Riemannian metric on $\mathbb{S}^{2n+1}$. The Hopf vector field $Z \in \mathfrak{X}(\mathbb{S}^{2n+1})$, given by $z \mapsto \mathbf{i}z$, satisfies $g_u(Z, Z) = 1$ and

$$(\mathcal{L}_Z g_u)(U, V) = Z(u)\; g_u(U, V), \quad U, V \perp Z.$$

Therefore Z is orthogonally conformal on every $(\mathbb{S}^{2n+1}, g_u)$. It can be deduced that Z is not conformal whenever $Z(u) \neq 0$. Note that in this family of examples the distribution $Z^\perp$ is not integrable.

Example 6. Even more, we will construct a compact Riemannian manifold which admits an orthogonally conformal vector field but does not admit a nowhere zero conformal one. We consider an $n(\geq 3)$-dimensional compact Riemannian manifold (M, g) whose Ricci tensor is negative definite (recall that on any $n(\geq 3)$-dimensional compact manifold M, there always exists a Riemannian metric g with everywhere negative definite Ricci tensor [1].) Next, we construct the compact manifold $\widetilde{M} = \mathbb{S}^1 \times M$ endowed with a twisted metric $\tilde{g} = \pi^*_{\mathbb{S}^1} d\theta^2 + f^2 \pi^*_M g$ and suppose that $\partial_\theta(\log(f))$ is not constant along each slice $\{e^{\mathbf{i}\theta}\} \times M$. As Example 4 shows, ∂_θ is orthogonally conformal and cannot be obtained by normalizing a nowhere zero conformal vector field. Let $\tilde{K} \in \mathfrak{X}(\widetilde{M})$ be a conformal vector field with $\mathcal{L}_{\tilde{K}}\tilde{g} = 2\rho\,\tilde{g}$. Put $K = \tilde{K} - \tilde{g}(\tilde{K}, \partial_\theta)\partial_\theta$, then we have $K_{(e^{\mathbf{i}\theta}, p)} \in T_pM \subset T_{(e^{\mathbf{i}\theta}, p)}\widetilde{M}$ and

$$(\mathcal{L}_K \tilde{g})(U, V) = 2\sigma\,\tilde{g}(U, V),$$

for any $U, V \perp \partial_\theta$, where $\sigma = \rho - \tilde{g}(\tilde{K}, \partial_\theta)\partial_\theta(\log(f))$. Therefore, for every θ we deduce that the vector field $K_\theta \in \mathfrak{X}(M)$ defined by $(K_\theta)_p = K_{(e^{i\theta},p)}$ is conformal. From our assumption on the Ricci tensor of (M, g) and as a consequence of the classical result by H. Wu [7] we obtain that $K_\theta = 0$ for every θ. Therefore $K = 0$, which means that $\tilde{K}$ must be proportional to ∂_θ. In case $\tilde{K}$ has no zero, then ∂_θ should be its normalization.

Example 7. Let $\pi : (M, g) \longrightarrow (B, g')$ be a Riemannian submersion with 1-dimensional fibers. The fundamental tensor fields of π (O'Neill tensors) are given by

$$T_E F = h\Big(\nabla_{v(E)} v(F)\Big) + v\Big(\nabla_{v(E)} h(F)\Big)$$
$$A_E F = v\Big(\nabla_{h(E)} h(F)\Big) + h\Big(\nabla_{h(E)} v(F)\Big),$$

where $E, F \in \mathfrak{X}(M)$ and v (resp. h) denotes the vertical (resp. horizontal) projection [2]. A unit vertical vector field $W \in \mathfrak{X}(M)$ is orthogonally Killing. In fact, let $X, Y \in \mathfrak{X}(M)$ be horizontal vector fields then,

$$(\mathcal{L}_W g)(X, Y) = -g(A_X Y, W) - g(A_Y X, W) = 0,$$

since the fundamental tensor A of π satisfies $A_X Y = -A_Y X$, [2]. Note that W is a Killing vector field if and only if the fundamental tensor T vanishes identically, that is, the fibers are totally geodesics.

3 Proofs of the results

We begin this section by recalling a formula which is our main tool. Let us remark that our approach has been inspired from the recent application of Bochner technique to Lorentzian Geometry [4], [5] and [6]. Let (M, g) be a Riemannian manifold and let X be any vector field on M. If ∇ denotes the Levi-Civita connection of g, then we have a (1,1)-tensor field $\mathbf{A}_X$ on M given by $\mathbf{A}_X(v) = -\nabla_v X$, for any $v \in T_pM$, $p \in M$. Note that trace $\mathbf{A}_X = -\mathrm{div}(X)$. If M is compact, then the following integral formula holds true,

$$\int_M \Big\{\mathrm{Ric}(X, X) + \mathrm{trace}(\mathbf{A}_X^2) - (\mathrm{trace}\,\mathbf{A}_X)^2\Big\}\, d\mu_g = 0. \tag{2}$$

Proof. Theorem 1 If Z is an orthogonally conformal vector field, we have

$$g(\mathbf{A}_Z U, V) + g(U, \mathbf{A}_Z V) = -2\rho g(U, V), \quad U, V \perp Z. \tag{3}$$

Direct computations show

$$\mathrm{trace}\,\mathbf{A}_Z = -\rho(n-1) \text{ and } \mathrm{trace}(\mathbf{A}_Z^2) = 2\rho^2(n-1) - \|\nabla Z\|^2 + \|\nabla_Z Z\|^2. \tag{4}$$

Observe that $\|\nabla Z\|^2 \geq \|\nabla_Z Z\|^2$, and equality holds if, and only if, $\nabla_U Z = 0$ for any $U \perp Z$, and in this case $\rho = 0$. Now, making use of (4) in (2) we get

$$\int_M \mathrm{Ric}(Z, Z) d\mu_g = (n-1)(n-3)\int_M \rho^2 d\mu_g + \int_M \Big\{\|\nabla Z\|^2 - \|\nabla_Z Z\|^2\Big\}\, d\mu_g, \tag{5}$$

which concludes the proof. ∎

Remark 2. If a unit vector field Z on a Riemannian manifold (M, g) satisfies $\nabla_U Z = 0$ for any $U \perp Z$ (hence Z is orthogonally Killing) then the distribution $Z^{\perp}$ is integrable with totally geodesic leaves.

Remark 3. When $\dim M = 2$, $\mathrm{trace}(\mathbf{A}_Z^2) = (\mathrm{trace}\,\mathbf{A}_Z)^2$ and $\mathrm{Ric}(Z, Z) = \mathcal{K}$, the Gaussian curvature of (M, g), for any unit vector field Z on (M, g). Hence, in the compact case, equation (2) reduces to $\int_{\mathbb{T}^2} \mathcal{K} d\mu_g = 0$, which is the well known Gauss-Bonnet theorem for a Riemannian metric on the torus $\mathbb{T}^2$.

Proof. Corollary 1 If (M, g) admits an orthogonally conformal vector field Z, from Theorem 1 we get $0 \leq \int_M \mathrm{Ric}(Z, Z) d\mu_g = \lambda \mathrm{vol}(M, g)$, and therefore $\lambda \geq 0$. ■

Proof. Corollary 2 Suppose that (M, g) admits an orthogonally conformal vector field Z. From Theorem 1 and the assumption on the Ricci tensor, we deduce that $\mathrm{Ric}(Z, Z) = 0$ on the whole M. Finally, being the Ricci tensor negative definite at a point $p \in M$, we obtain $Z_p = 0$ which is a contradiction. ■

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References

[1] J. Lohkamp, *Metrics of negative Ricci curvature*, Ann. Math. (2) **140** (1994), 655–683.

[2] B. O'Neill, The fundamental equations of a submersion, *Mich. Math. J.* **13** (1966), 459–469.

[3] B. O'Neill, *Semi-Riemannian Geometry with applications to Relativity*, Academic Press, 1983.

[4] A. Romero, *The introduction of Bochner's technique on Lorentzian manifolds*, Nonlinear Anal. **47** (2001), 3047–3059.

[5] A. Romero and M. Sánchez, *An integral inequality on compact Lorentz manifolds and its applications*, Bull. London Math. Soc. **28** (1996), 509–513.

[6] A. Romero and M. Sánchez, *Bochner's technique on Lorentzian manifolds and infinitesimal conformal symmetries*, Pacific J. Math. **186** (1998), 141–148.

[7] H. Wu, *A remark on the Bochner's technique in Differential Geometry*, Proc. Amer. Math. Soc. **78** (1980), 403–408.

[8] K. Yano, *On harmonic and Killing vector fields*, Ann. Math. (2) **55** (1952), 328–341.

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