A finite axiom scheme for approach frames

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Abstract

The theory of approach spaces [5, 6] has set the context in which numerical topological concepts exist. The successful interaction between frames and topology on the one hand [4] and the search for a good notion of sobriety in the context of approach theory on the other hand was the motivation to develop a theory of approach frames [1].

The original definition of approach frames was given in terms of an implicitly defined set of equations. In this work, we describe a subset of this by a finite axiom scheme (of only six types of equations) which implies all the equations originally involved and hence provides a substantial simplification of the definition of approach frames. Furthermore we show that the category of approach frames is the Eilenberg-Moore category for the monad determined by the functor which takes each approach frame to the set of its regular functions.

Introduction and background

An approach frame is a frame equipped with two families of unary operations, $(A_\alpha)_{\alpha\in[0,\infty]}$ and $(S_\alpha)_{\alpha\in[0,\infty]}$, which satisfy all identities that hold for $\wedge, \bigvee, 0, \infty, A_\alpha$, S_α in the frame $[0,\infty]$ with

$$A_\alpha : \lambda \mapsto \lambda + \alpha \text{ and } S_\alpha : \lambda \mapsto (\lambda - \alpha) \vee 0.$$

In general, the A_α and S_α are called shift operators, and specifically the up-shift and the down-shift operators, respectively. Also, at times, we shall use a unified

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notation and define O_α for each $\alpha \in \mathbb{R}$ as

$$O_\alpha := \begin{cases} A_\alpha & (\alpha \geq 0) \\ S_{-\alpha} & (\alpha < 0). \end{cases}$$

A morphism between approach frames is a shift operator preserving frame homomorphism.

In order to obtain the desired simplified axiomatic presentation of the category **AFrm** of approach frames, we introduce a finitely described subset $\mathcal{F}$ of the equations defining approach frames and then prove that any frame equipped with unary operations A_α and S_α, $\alpha \in [0, \infty]$, which satisfy the equations in $\mathcal{F}$ is in fact an approach frame. The technique for this is to show that any frame of the latter kind is the image of some approach frame by some shift operator-preserving frame homomorphism and hence itself an approach frame.

Approach spaces have applications in approximation theory [7, 8], probability theory [6] chapter 5.1, functional analysis [9, 14, 15, 16], categorical topology [3, 6] and many other fields. They have several well known equivalent characterizations [6]. One that is of particular interest to us is the characterization in terms of the regular function frame. Thus an approach space is a pair $(X, \mathcal{R})$ where $\mathcal{R} \subset [0, \infty]^X$ is a collection satisfying

(R1) $\forall A \subset \mathcal{R} : \ \bigvee_{\varphi \in A} \varphi \in \mathcal{R}$,

(R2) $\forall \varphi, \psi \in \mathcal{R} : \ \varphi \wedge \psi \in \mathcal{R}$,

(R3) $\forall \alpha \in \mathbb{R}^+, \forall \varphi \in \mathcal{R} : \varphi + \alpha \in \mathcal{R}$,

(R4) $\forall \alpha \in \mathbb{R}^+, \forall \varphi \in \mathcal{R} : (\varphi - \alpha) \vee 0 \in \mathcal{R}$.

A map $f : X \to Y$ is a contraction between approach spaces $(X, \mathcal{R}_X)$ and $(Y, \mathcal{R}_Y)$ if and only if $\mathcal{R}f(\varphi) := \varphi \circ f$ is in $\mathcal{R}_X$ whenever φ is in $\mathcal{R}_Y$. The category of approach spaces is denoted **Ap**.

Approach spaces form a topological construct [10] in which the constructs of metric spaces (with contractions) and of topological spaces are nicely embedded. Given a source $(f_i : X \to (X_i, \mathcal{R}X_i))_{i \in I}$, the initial regular function frame is given by

$$\mathcal{R}X = \{\mu_j \circ f_j | j \in I, \mu_j \in \mathcal{R}X_j\}^{\wedge \vee}, \tag{1}$$

with $\mathcal{S}^\wedge$ and $\mathcal{S}^\vee$ respectively the saturation of $\mathcal{S}$ with respect to finite meets and arbitrary joins [6].

Because the join and meet operation as well as subtraction $S_\alpha \varphi := \varphi \ominus \alpha$ with $\beta \ominus \alpha := (\beta - \alpha) \vee 0$ and addition $A_\alpha \varphi := \varphi + \alpha$ are defined in a pointwise way, any regular function frame is an approach frame. The resulting approach frame is denoted $\mathcal{R}_{\mathrm{f}} X$. We will also use the fact that **Ap** has an initially dense object $\mathbb{P} = ([0, \infty], \mathcal{R}_{\mathbb{P}})$ with

$$\mathcal{R}_{\mathbb{P}} = \{\varphi \in [0, \infty]^{[0,\infty]} | \forall x \in [0, \infty], \forall A \subset [0, \infty] : \varphi(x) - \sup_{a \in A} \varphi(a) \leq x - \sup A\}, \tag{2}$$

see e.g. [6].

1 A basic set of equations

We start by describing, in terms of a finite axiom scheme, a specific part of the equations used to define approach frames.

Definition 1.1. Let us put **OFrm** for the category of frames with arbitrary families $(A_\alpha)_{\alpha\in[0,\infty]}$ and $(S_\alpha)_{\alpha\in[0,\infty]}$ of unary operations and the frame homomorphisms that commute with each of these as morphisms. For each α, β in $[0,\infty]$ and for each subset $Y \subset [0,\infty]$ we consider the following formulas:

(I1) $A_\alpha \circ A_\beta = A_{\alpha+\beta}$,

(I2) $\forall a : S_\alpha A_\alpha a = a$ for all ($\alpha < \infty$),

(I3) $\forall a : S_\infty a = \bot$,

(I4) $\forall a : A_\alpha S_\alpha a = a \vee A_\alpha \bot$,

(I5) $\forall a, b : A_\alpha(a \vee b) = A_\alpha a \vee A_\alpha b$,

(I6) $\forall a : \bigvee_{\alpha\in Y} A_\alpha a = A_{\sup Y} a$.

Let $\mathcal{F}$ be the set of all equations (I1),...,(I6), where α, β runs through $[0,\infty]$ and Y runs through $2^{[0,\infty]}$, and let **Mod**$\mathcal{F}$ be the subcategory of those $L \in$ **OFrm** which satisfies these identities.

We denote the forgetful functor from **Mod**$\mathcal{F}$ to **Set** by U and we use the notation $\boldsymbol{\lambda}$ for $A_\lambda \bot$.

Since $\mathcal{F}$ is a subset of the equation-set defining approach frames, it is clear that

$$\mathbf{AFrm} \subset \mathbf{Mod}\mathcal{F}.$$

In the next section we show that both categories are equal. To this end we derive some further approach frame identities from those given in 1.1.

Proposition 1.2. Given any $L \in$ **Mod**$\mathcal{F}$, the following identities hold for all $\alpha, \beta \in [0,\infty]$, and for all $a, b \in L$, $X \subset L$, $Y \subset [0,\infty]$.

rule(1) $A_\infty a = \top$,

rule(2) $A_0 a = a$,

rule(3) $S_0 a = a$,

rule(4) $S_{\alpha+\beta} = S_\alpha \circ S_\beta$,

rule(5) $S_\alpha(\bigvee X = \bigvee_{a\in X} S_\alpha a$,

rule(6) $A_\alpha(a \wedge b) = A_\alpha a \wedge A_\alpha b$,

rule(7) $A_\alpha(\bigvee X) = \bigvee_{a\in X} A_\alpha x$ for $X \neq \emptyset$,

rule(8) $S_\alpha(a \wedge b) = S_\alpha a \wedge S_\alpha b$,

rule(9) $a \leq A_\alpha a$,

rule(10) $\bigvee_{\alpha\in Y} S_\alpha a = S_{\inf Y} a$,

rule(11) $S_\alpha a \leq a$,

rule(12) $A_\alpha a \wedge A_\beta a = A_{\alpha\wedge\beta} a$,

rule(13) $S_\alpha a \wedge S_\beta a = S_{\alpha\vee\beta} a$,

rule(14) $O_\alpha a \leq O_\beta a$ if $\alpha \leq \beta$.

Proof.

(1) By (I3) and (I4) we have for all $a \in L$ that $A_\infty \bot = A_\infty S_\infty a = a \vee A_\infty \bot$, hence $A_\infty \bot = \top$. Using (I5), we then conclude that $A_\infty a = \top$ for all $a \in L$.

(2) Note that by (I1) $A_0 a = A_{0+0} a = A_0 A_0 a$, so applying S_0 and (I2) yields the desired result.

(3) From the previous result and from (I2) we obtain $S_0 a = S_0 A_0 a = a$.

(4) For $\alpha, \beta < \infty$, we have by (I1), (I4) and (I5) that

$$\begin{aligned} A_{\alpha+\beta} S_\alpha S_\beta a &= A_\beta A_\alpha S_\alpha S_\beta a = A_\beta (S_\beta a \vee \boldsymbol{\alpha}) = a \vee \boldsymbol{\beta} \vee (\boldsymbol{\alpha} + \boldsymbol{\beta}) \\ &= a \vee (\boldsymbol{\alpha} + \boldsymbol{\beta}) = A_{\alpha+\beta} S_{\alpha+\beta} a \end{aligned}$$

and applying $S_{\alpha+\beta}$ and (I2) gives the wanted result.
If $\alpha = \infty$ the result is immediate and if $\beta = \infty$ we use the easily seen fact that $S_\alpha \mathbf{0} = \mathbf{0}$.

(5) It suffices to show that $S_\alpha(a \vee b) = S_\alpha a \vee S_\alpha b$. This shows that S_α is order-preserving, we have that A_α has this property by (I5) and combined with (I2) and (I4) this gives us that A_α is the right Galois adjoint of S_α for all α. Since we are working in a complete lattice, we then have that S_α commutes with all joins and A_α with all meets.

For $\alpha = \infty$, the result is immediate. For $\alpha < \infty$ use (I2), (I4) and (I5) to find

$$\begin{aligned} S_\alpha (a \vee b) &= S_\alpha A_\alpha S_\alpha (a \vee b) = S_\alpha (a \vee b \vee \boldsymbol{\alpha}) = S_\alpha ((a \vee \boldsymbol{\alpha}) \vee (b \vee \boldsymbol{\alpha})) \\ &= S_\alpha (A_\alpha S_\alpha a \vee A_\alpha S_\alpha b) = S_\alpha A_\alpha (S_\alpha a \vee S_\alpha b) = S_\alpha a \vee S_\alpha b \end{aligned}$$

(6) Immediate consequence of the proof of the previous rule.

(7) For $\alpha = \infty$, we have nothing to prove. For $\alpha < \infty$ we use (I2), rule(5) and the fact that A_α is order-preserving, hence $A_\alpha a \vee \boldsymbol{\alpha} = A_\alpha a$ and obtain

$$A_\alpha \left(\bigvee X \right) = A_\alpha \left(\bigvee_{a \in X} (S_\alpha A_\alpha a) \right) = A_\alpha S_\alpha \left(\bigvee_{a \in X} A_\alpha a \right) = \bigvee_{a \in X} A_\alpha a$$

(8) This is shown analogous to the proof of rule(5), using distributivity and rule(6) instead of (I5).

(9) Immediate from rule(2) and (I6).

(10) First, remark that $S_\infty a \leq S_\alpha a$ for all α by (I3). Then, for $\alpha, \beta \in \mathbb{R}^+$ with $\alpha < \beta$ we have by (I1), (I4), (I5) and rule(9)

$$\begin{aligned} A_\beta (S_\alpha a \vee S_\beta a) &= A_\beta S_\alpha a \vee A_\beta S_\beta a = A_{\beta-\alpha} A_\alpha S_\alpha a \vee A_\beta S_\beta a \\ &= A_{\beta-\alpha}(a \vee \boldsymbol{\alpha}) \vee (a \vee \boldsymbol{\beta}) = A_{\beta-\alpha} a \vee a \vee \boldsymbol{\beta} \\ &= A_{\beta-\alpha} a \vee \boldsymbol{\beta} = A_{\beta-\alpha}(a \vee \boldsymbol{\alpha}) = A_{\beta-\alpha} A_\alpha S_\alpha a = A_\beta S_\alpha a. \end{aligned}$$

Applying S_β and (I2) then gives the identity $S_\alpha a \vee S_\beta a = S_\alpha a$.
Now take an arbitrary set $Y \subset \mathbb{R}^+$. By our remarks above, we can limit

ourselves to a bounded set Y. Take $\beta = \sup_{i\in I} \alpha_i$, using (I1), (I4) and rule(7), we see

$$\begin{aligned} A_\beta\left(\bigvee_{\alpha\in Y} S_\alpha a\right) &= \bigvee_{\alpha\in Y} A_{\beta-\alpha} a \vee \boldsymbol{\beta} = A_{\sup(\beta-\alpha)} a \vee \boldsymbol{\beta} \\ &= A_{\beta-\inf Y} a \vee \boldsymbol{\beta} = A_\beta S_{\inf Y} a \end{aligned}$$

and by (I2) we obtain $\bigvee_{\alpha\in Y} S_\alpha a = S_{\inf Y} a$.

(11) Immediate from rule(3) and rule(10).

(12) (I6) implies $A_\alpha a \leq A_\beta a$ whenever $\alpha \leq \beta$ and hence also $A_\alpha a \wedge A_\beta a = A_\alpha a = A_{\alpha\wedge\beta} a$ which proves the point since $[0,\infty]$ is totally ordered.

(13) Analogous to the previous rule, using rule(10) instead of (I6).

(14) If $\alpha \leq 0 \leq \beta$, use rule(9) and rule(11). For $\alpha, \beta \geq 0$, we use (I6) and given $\alpha, \beta \leq 0$ we need rule(10). ∎

Note that there are dependencies between these derived rules and the axioms, for example given (I1)–(I4) and the finite version of rule(5) we can prove (I5). Furthermore, with (I1)–(I5) given, we can show that rule(10) implies (I6) for collections $(\alpha_i)_{i\in I}$ with $\sup_{i\in I} \alpha_i < \infty$.

Remark also that by rule(5) we have that rule(10) is equivalent to rule(10) formulated for bounded sets $Y \subset [0,\infty]$ with $\inf Y = 0$.

2 Main result

Theorem 2.1. For any set S, the approach frame $\mathcal{R}_\mathrm{f}\left(\mathbb{P}^S\right)$ is free on $\{\mathrm{ev}(\cdot,s) \mid s \in S\}$ in the category $\mathbf{Mod}\mathcal{F}$: for any set map $\tau : S \to UL$, $L \in \mathbf{Mod}\mathcal{F}$, there exists a unique $h : \mathcal{R}_\mathrm{f}\left(\mathbb{P}^S\right) \to L$ in $\mathbf{Mod}\mathcal{F}$ such that

$$\forall s \in S : h(\mathrm{ev}(\cdot,s)) = \tau(s).$$

Proof. Any $\varphi \in \mathcal{R}_\mathbb{P}$ can be expressed as $\bigvee_{x\in[0,\infty[} \left((\mathrm{Id} + (\varphi(x) - x)) \vee 0\right) \wedge \varphi(x)$, so

$$\mathcal{R}_\mathbb{P} = \{(\mathrm{Id} \overset{+}{\ominus} \alpha) \wedge \lambda \mid \alpha, \lambda \in \mathbb{R}^+\}^{\wedge\bigvee}$$

and hence from (1) we find the following formula

$$\mathcal{R}_\mathrm{f}\left(\mathbb{P}^S\right) = \left\{(\mathrm{ev}(\cdot,s) \overset{+}{\ominus} \alpha) \wedge \lambda \;\middle|\; s \in S,\ \alpha, \lambda \in \mathbb{R}^+\right\}^{\wedge\bigvee}. \tag{3}$$

Let f be in $\mathcal{R}_\mathrm{f}\left(\mathbb{P}^S\right)$. By (3) there exists a collection $Y \subset [0,\infty]$, a collection $(K_\lambda)_{\lambda\in Y}$ of finite subsets of S, and for each K_λ a collection $(\alpha_s)_{s\in K_\lambda} \subset \mathbb{R}$, such that

$$f = \bigvee_{\lambda\in Y} \bigwedge_{s\in K_\lambda} O_{\alpha_s} \mathrm{ev}(\cdot,s) \wedge \lambda.$$

It suffices to show that the assignment

$$h : \mathcal{R}_{\mathrm{f}}\left(\mathbb{P}^S\right) \to L : \bigvee_{\lambda \in Y} \bigwedge_{s \in K_\lambda} O_{\alpha_s}\mathrm{ev}(\cdot, s) \wedge \lambda \mapsto \bigvee_{\lambda \in Y} \bigwedge_{s \in K_\lambda} O_{\alpha_s}\tau(s) \wedge \boldsymbol{\lambda}$$

is a well defined $\mathbf{Mod}\mathcal{F}$-morphism, uniqueness of h then follows from (3).

We first show that h is well defined, that is,

$$\bigvee_{\lambda \in Y} \bigwedge_{s \in K_\lambda} O_{\alpha_s}\tau(s) \wedge \boldsymbol{\lambda} = \bigvee_{\mu \in Z} \bigwedge_{t \in M_\mu} O_{\beta_t}\tau(t) \wedge \boldsymbol{\mu}$$

whenever

$$\bigvee_{\lambda \in Y} \bigwedge_{s \in K_\lambda} O_{\alpha_s}\mathrm{ev}(\cdot, s) \wedge \lambda = \bigvee_{\mu \in Z} \bigwedge_{t \in M_\mu} O_{\beta_t}\mathrm{ev}(\cdot, t) \wedge \mu.$$

So the desired result follows by symmetry if we show that

$$\bigwedge_{s \in K} O_{\alpha_s}\tau(s) \wedge \boldsymbol{\lambda} \leq \bigvee_{\mu \in Z} \bigwedge_{t \in M_\mu} O_{\beta_t}\tau(t) \wedge \boldsymbol{\mu}$$

for $\lambda \in [0, \infty]$, K a finite subset of S and $(\alpha_s)_{s \in S}$ a collection in $[0, \infty]$ such that

$$\bigwedge_{s \in K} O_{\alpha_s}\mathrm{ev}(\cdot, s) \wedge \lambda \leq \bigvee_{\mu \in Z} \bigwedge_{t \in M_\mu} O_{\beta_t}\mathrm{ev}(\cdot, t) \wedge \mu. \tag{4}$$

We can suppose that $\alpha_s \leq \lambda$ for each $s \in K$, since if $\alpha_s > \lambda$, then also $O_{\alpha_s}\mathrm{ev}(\cdot, s) \geq \lambda$. We can also suppose that $\lambda < \infty$, since for $\lambda = \infty$ we use that $f = \bigvee_{n \in \mathbb{N}} f \wedge n$ and the same goes in L by (I3) and (I6). Take now $K_+ := \{s \in K | \alpha_s \geq 0\}$ and $K_- := K \setminus K_+$, then evaluating (4) in $\varphi \in [0, \infty]^S$ given by

$$\varphi(s) := \begin{cases} \lambda - \alpha_s & s \in K_+, \\ \lambda + \alpha_s & s \in K_-, \\ 0 & \text{otherwise}, \end{cases}$$

we obtain

$$\lambda \leq \sup_{\mu \in Z} \inf_{t \in M_\mu} (O_{\beta_t}\varphi(t)) \wedge \mu.$$

For each $\epsilon \in]0, \lambda[$, take $\mu_\epsilon \in Z$ such that

$$\lambda - \epsilon \leq \inf_{t \in M_{\mu_\epsilon}} (O_{\beta_t}\varphi(t)) \wedge \mu_\epsilon. \tag{5}$$

For each ϵ we make the decomposition $M_{\mu_\epsilon} = (M_{\mu_\epsilon} \setminus K) \cup M^+_{\mu_\epsilon} \cup M^-_{\mu_\epsilon}$, with $M^+_{\mu_\epsilon} := M_{\mu_\epsilon} \cap K_+$ and $M^-_{\mu_\epsilon} := M_{\mu_\epsilon} \cap K_-$. Thus the inequality (5) falls apart:

$$\lambda - \epsilon \leq \min_{t \in M^+_{\mu_\epsilon}} O_{\beta_t}(\lambda - \alpha_t), \tag{6}$$

$$\lambda - \epsilon \leq \min_{t \in M^-_{\mu_\epsilon}} O_{\beta_t}(\lambda + \alpha_t), \tag{7}$$

$$\lambda - \epsilon \leq \min_{t \in M_{\mu_\epsilon} \setminus K} O_{\beta_t}(0), \tag{8}$$

$$\lambda - \epsilon \leq \mu_\epsilon. \tag{9}$$

Explicitly inequality (6) looks like

$$\lambda - \epsilon \leq \min_{t \in M_1}(\lambda - \alpha_t + \beta_t) \wedge \min_{t \in M_2}(((\lambda - \alpha_t) - |\beta_t|) \vee 0),$$

with $M_1 := \{t \in M_{\mu_\epsilon} \mid \beta_t \geq 0\}$ and $M_2 := \{t \in M_{\mu_\epsilon} \mid \beta_t < 0\}$. Thus for $t \in M_1$ we have $\alpha_t - \epsilon \leq \beta_t$ and so $S_\epsilon A_{\alpha_t}\tau(t) \leq A_{\beta_t}\tau(t)$ by rule(14) (since $S_\epsilon A_{\alpha_t} = O_{\alpha_t - \epsilon}$ by (I1), (I2) and rule(4)). For $t \in M_2$ we formally have $\lambda - \epsilon \leq (\lambda - \alpha_t - \beta_t) \vee 0$, which actually reduces to $\lambda - \epsilon \leq \lambda - \alpha_t - \beta_t$, thus $\alpha_t - \epsilon \leq -\beta_t$, so analogously we find $S_\epsilon A_{\alpha_t}\tau(t) \leq S_\beta \tau(t)$. By rule(8) we then obtain

$$S_\epsilon \bigwedge_{s \in K_+} A_{\alpha_s}\tau(s) = \bigwedge_{s \in K_+} S_\epsilon A_{\alpha_s}\tau(s) \leq \bigwedge_{t \in M^+_{\mu_\epsilon}} O_{\beta_t}\tau(t).$$

We decompose (7) analogously and for $t \in M_1$ we have $S_{\alpha_t + \epsilon} t \leq A_{\beta_t}\tau(t)$ and for $t \in M_2$ we find $S_{\alpha_t+\epsilon}\tau(t) \leq S_{\beta_t}\tau(t)$ by rule(14). Using rule(8) and rule(4) we get

$$S_\epsilon \bigwedge_{s \in K_-} S_{\alpha_s}\tau(s) = \bigwedge_{s \in K_-} S_{\alpha_s + \epsilon}\tau(s) \leq \bigwedge_{s \in K_-} O_{\beta_s}\tau(s) \leq \bigwedge_{t \in M^-_{\mu_\epsilon}} O_{\beta_t}\tau(t).$$

From (8) we deduce in a similar way that

$$\boldsymbol{\lambda} - \boldsymbol{\epsilon} \leq \bigwedge_{t \in M_{\mu_\epsilon} \setminus K} O_{\beta_t}\tau(t).$$

Finally, from (9) we know by rule(14)

$$\boldsymbol{\lambda} - \boldsymbol{\epsilon} \leq \boldsymbol{\mu_\epsilon}.$$

Using rule(3), rule(10) and rule(8), we find

$$\begin{aligned}
\bigwedge_{s \in K_+} A_{\alpha_s}\tau(s) \wedge \bigwedge_{s \in K_-} S_{\alpha_s}\tau(s) \wedge \boldsymbol{\lambda} &= \bigvee_{\epsilon > 0} S_\epsilon \left(\bigwedge_{s \in K_+} A_{\alpha_s}\tau(s) \wedge \bigwedge_{s \in K_-} S_{\alpha_s}\tau(s) \wedge \boldsymbol{\lambda} \right) \\
&= \bigvee_{\epsilon > 0} S_\epsilon(\bigwedge_{s \in K_+} A_{\alpha_s}\tau(s)) \wedge S_\epsilon(\bigwedge_{s \in K_-} S_{\alpha_s}\tau(s)) \wedge \boldsymbol{\lambda} - \boldsymbol{\epsilon} \\
&\leq \bigvee_{\epsilon > 0} \bigwedge_{t \in M^+_{\mu_\epsilon}} O_{\beta_t}\tau(t) \wedge \bigwedge_{t \in M^-_{\mu_\epsilon}} O_{\beta_t}\tau(t) \wedge \bigwedge_{t \in M_{\mu_\epsilon} \setminus K} O_{\beta_t}\tau(t) \wedge \boldsymbol{\mu_\epsilon} \\
&\leq \bigvee_{\epsilon > 0} \bigwedge_{t \in M_{\mu_\epsilon}} O_{\beta_t}\tau(t) \wedge \boldsymbol{\mu_\epsilon} \leq \bigvee_{\mu \in Z} \bigwedge_{t \in M_\mu} O_{\beta_t}\tau(t) \wedge \boldsymbol{\mu}.
\end{aligned}$$

To show that h is a $\mathbf{Mod}\mathcal{F}$-morphism, we will first remark that it is a frame homomorphism: by construction it is clear that h commutes with arbitrary joins and, since $\mathcal{R}_{\mathrm{f}}\left(\mathbb{P}^S\right)$ and L are frames, h also commutes with finite meets. We have that h commutes with A_∞ and S_∞ by (I3). Take $\alpha < \infty$. To see that

$$\begin{aligned}
h\Big(A_\alpha(\bigvee_{\lambda \in Y} \bigwedge_{s \in K_\lambda} O_{\alpha_s}\mathrm{ev}(\cdot, s) \wedge \lambda)\Big) &= h\Big(\bigvee_{\lambda \in Y} (\bigwedge_{s \in K_\lambda} O_{\alpha_s + \alpha}\mathrm{ev}(\cdot, s) \wedge (\lambda + \alpha)) \vee \alpha\Big) \\
&= \bigvee_{\lambda \in Y} (\bigwedge_{s \in K_\lambda} O_{\alpha_s + \alpha}\tau(s) \wedge (\boldsymbol{\lambda} + \boldsymbol{\alpha})) \vee \boldsymbol{\alpha} = A_\alpha(\bigvee_{\lambda \in Y} \bigwedge_{s \in K_\lambda} O_{\alpha_s}\tau(s) \wedge \boldsymbol{\lambda})
\end{aligned}$$

we use that $A_\alpha O_\beta = O_{\alpha+\beta} \vee \boldsymbol{\alpha}$ (which follows out (I1), (I2), (I4) and rule(4)), rule(7) and rule(6). For S_α, the proof is analogous with the application of $S_\alpha O_\beta = O_{\beta-\alpha}$, rule(5) and rule(8). ∎

Corollary 2.2. Any $L \in \mathbf{Mod}\mathcal{F}$ is the homomorphic image of an approach frame.

Proof. Take an onto map $\tau : S \to UL$ and apply the previous theorem. ∎

Using that **AFrm** is equational and that homomorphisms transport algebraic equalities we can conclude the following theorem.

Theorem 2.3. The categories $\mathbf{Mod}\mathcal{F}$ and **AFrm** are equal. Hence, in order to prove that a frame L with operation A_λ and S_λ is an approach frame, it suffices to check the equalities in 1.1.

Corollary 2.4. The functor $\mathcal{R}_{\mathrm{f}}(\mathbb{P}^-) : \mathbf{Set} \to \mathbf{AFrm}$ is left adjoint to the underlying set functor $\mathrm{U} : \mathbf{AFrm} \to \mathbf{Set}$.

3 AFrm as Eilenberg-Moore category derived from approach spaces

Note that we have a functor

$$\mathcal{R} : \mathbf{Ap}^{\mathrm{op}} \to \mathbf{Set} : (X \xrightarrow{f} Y) \longmapsto (\mathcal{R}Y \xrightarrow{\mathcal{R}f} \mathcal{R}X),$$

with $\mathcal{R}f(\phi) = \phi \circ f$. It follows from the fat that $\mathcal{R} = \mathbf{Ap}(-, \mathbb{P})$ that the assignment

$$\mathbb{P}^- : \mathbf{Set} \to \mathbf{Ap}^{\mathrm{op}} : (S_1 \xrightarrow{f} S_2) \longmapsto (\mathbb{P}^{S_2} \xrightarrow{\mathbb{P}^-(f)} \mathbb{P}^{S_1}),$$

with $\mathbb{P}^-(f)(a) := a \circ f$ for $a : S_2 \to [0, \infty]$, introduces a functor that is left adjoint to $\mathcal{R}$.

Since $\mathcal{R}f$ is a shift operator preserving frame homomorphism, the functor $\mathcal{R}$ extends to

$$\mathcal{R}_{\mathrm{f}} : \mathbf{Ap}^{\mathrm{op}} \to \mathbf{AFrm} : (X \xrightarrow{f} Y) \longmapsto (\mathcal{R}_{\mathrm{f}}Y \xrightarrow{\mathcal{R}f} \mathcal{R}_{\mathrm{f}}X).$$

Theorem 3.1. The category **AFrm** is monadic. Moreover, in the following diagram

$$\begin{tikzcd}
\mathbf{Ap}^{\mathrm{op}} \arrow[r, "\mathcal{R}_{\mathrm{f}}"] \arrow[d, shift right, "\mathcal{R}"'] \arrow[d, phantom, "\vdash"] & \mathbf{AFrm} \arrow[d, shift right, "\mathrm{U}"'] \arrow[d, phantom, "\vdash"] \\
\mathbf{Set} \arrow[u, shift right, "\mathbb{P}^-"'] \arrow[r, equal] & \mathbf{Set}, \arrow[u, shift right, "\mathcal{R}_{\mathrm{f}}(\mathbb{P}^-)"']
\end{tikzcd} \tag{10}$$

$\mathcal{R}_{\mathrm{f}}$ is the comparison functor of the adjunction $\mathcal{R} \vdash \mathbb{P}^-$.

Proof. Note that the co-unit of the adjunction $\mathcal{R} \vdash \mathbb{P}^-$ is the map

$$\epsilon_X : X \to \mathbb{P}^{\mathcal{R}X} : x \mapsto \mathrm{ev}(\cdot, x),$$

so the monad is

$$\mathbb{T} = (\mathrm{T}, \eta, \mu), \tag{11}$$

with $\mathrm{T} = \mathcal{R}(\mathbb{P}^-)$ and where the multiplication is $\mu = \mathcal{R}\epsilon_{\mathbb{P}^-}$ [12], thus given on a set S by

$$\mu_S : \mathrm{T}^2\, S \to \mathrm{T}\, S : \psi \longmapsto \left(\mu_S(\psi) : [0,\infty]^S \to [0,\infty] : a \mapsto \psi(\mathrm{ev}(\cdot, a))\right).$$

Since **AFrm** is an equationally defined category with a left adjoint to the forgetful functor, it follows follows from [4, 12] that this category is monadic.

It is clear that both adjunctions $\mathcal{R}_\mathrm{f}(\mathbb{P}^-) \dashv \mathrm{U}$ and $\mathbb{P}^- \dashv \mathcal{R}$ have the same unit η. Let S be a set. The assignment

$$\mathbb{P}^S \to \mathbb{P}^{\mathcal{R}(\mathbb{P}^S)} : a \mapsto \mathrm{ev}(\cdot, a)$$

is a morphism in **Ap**. Hence, by applying $\mathcal{R}_\mathrm{f}$, we obtain the following morphism of approach frames

$$\mathcal{R}_\mathrm{f}\left(\mathbb{P}^{\mathcal{R}(\mathbb{P}^S)}\right) \to \mathcal{R}_\mathrm{f}\left(\mathbb{P}^S\right) : \psi \mapsto h(\psi),$$

with

$$h(\psi) : [0,\infty]^S \to [0,\infty] : a \mapsto \psi(\mathrm{ev}(\cdot, a)).$$

Note that for any $\phi \in \mathcal{R}\left(\mathbb{P}^S\right) = U\mathcal{R}_\mathrm{f}\left(\mathbb{P}^S\right)$,

$$h(\mathrm{ev}(\cdot, \phi)) = \phi,$$

so h is the counit of the adjunction $\mathcal{R}_\mathrm{f}(\mathbb{P}^-) \dashv \mathrm{U}$ and hence Uh the corresponding multiplication. This shows that both adjunctions have the same multiplication, and hence the same monad $\mathbb{T}$ (11).

The fact that $\mathcal{R}_\mathrm{f}$ is indeed the comparison functor follows from [11] (VI Theorem 1), because in the diagram (10) both the $\downarrow\rightrightarrows\downarrow$-square and the $\uparrow\rightrightarrows\uparrow$-square commute. ∎

With the previous result we can also compare the **AFrm** $-$ **Ap** and the **Frm** $-$ **Top** situations with each other. For the latter there is the diagram analogous to the one in (10)

$$\begin{tikzcd}
\mathbf{Top}^{\mathrm{op}} \arrow[r, "\mathcal{O}"] \arrow[d, shift right, "\mathbf{Top}(-,\mathbb{S})"'] \arrow[d, phantom, "\vdash"] & \mathbf{Frm} \arrow[d, shift right, "\mathrm{U}"'] \arrow[d, phantom, "\vdash"] \\
\mathbf{Set} \arrow[u, shift right, "\mathbb{S}^-"'] \arrow[r, equal] & \mathbf{Set}, \arrow[u, shift right, "\mathcal{O}\mathbb{S}^-"']
\end{tikzcd}$$

where $\mathbb{S}$ is the Sierpinski space, $\mathbb{S}^-$ is left adjoint to $\mathbf{Top}(-,\mathbb{S})$ as a general fact, $U\mathcal{O} \simeq \mathbf{Top}(-,\mathbb{S})$ by the nature of $\mathbb{S}$, and $\mathcal{O}\mathbb{S}^-$ is left adjoint to the underlying set functor U. Furthermore everything works out as in the previous case, so that **Frm** is seen as the Eilenberg-Moore category determined by $\mathbf{Top}(-,\mathbb{S})$ with $\mathcal{O}$ as the comparison functor.

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