Compositions of harmonic mappings and biharmonic mappings*

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Abstract

The aim of this paper is twofold. First, we investigate the properties of the composition of harmonic mappings with harmonic mappings, and the composition of biharmonic mappings with harmonic mappings. Second, we consider the Goodman-Saff conjecture for biharmonic mappings in the unit disk. In fact, we show that the answer to the Goodman-Saff conjecture is positive for a special class of univalently biharmonic mappings which contains the set of all harmonic univalent mappings.

1 Introduction and Preliminaries

A four times continuously differentiable complex-valued function $F = u + iv$ in a domain $D \subset \mathbb{C}$ is *biharmonic* if and only if ΔF, the Laplacian of F, is harmonic in D. Note that ΔF is harmonic in D if F satisfies the biharmonic equation $\Delta(\Delta F) = 0$ in D, where Δ represents the Laplacian operator

$$\Delta = 4\frac{\partial^2}{\partial z \partial \overline{z}}.$$

Biharmonic functions arise in a lot of physical situations, particularly in fluid dynamics and elasticity problems, and have many important applications in engineering and biology. See [7, 8, 9] for more details.

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It has been shown that a mapping F is biharmonic in a simply connected domain D if and only if F has the following representation

$$F = |z|^2 G + K,$$

where G and K are complex-valued harmonic functions in D (cf. [1, 2]). Also it has been known that G and K can be expressed as

$$G = g_1 + \overline{g_2} \text{ and } K = k_1 + \overline{k_2},$$

where g_1, g_2, k_1 and k_2 are analytic in D (cf. [3, 5]).

It is known that a harmonic mapping of an analytic function is harmonic, but an analytic function of a harmonic mapping is not necessarily harmonic (cf. [5]).

In 1987, Reich discussed the harmonicity of the composition of two harmonic mappings and obtained the following:

Theorem A. [10, Theorem 1] *Suppose $f(z) = z + \overline{B(z)}$ and $G(z) = B'(z)$, where $B(z)$ is analytic in the neighborhood under consideration. A necessary and sufficient condition that there locally exists a non-affine complex harmonic function $g(w)$, such that $g(f(z))$ is harmonic is that $G(z)$ satisfies*

$$G'^2 = \alpha^2 G^4 + 2cG^3 + \overline{\alpha}^2 G^2$$

for some complex constant α and some real constant c.

In [10], Reich have also included a proof of the following version of the Choquet-Deny theorem.

Theorem B. *Suppose f is a sense-preserving harmonic homeomorphism and is neither analytic nor affine. A necessary and sufficient condition that f^{-1} is also harmonic is that*

$$f(z) = \frac{\sigma}{\overline{\alpha}} z + \frac{1}{\overline{\alpha}} \log \left[\frac{\mu - e^{-\sigma z}}{\overline{\mu - e^{-\sigma z}}} \right] + const,$$

where σ, α, μ are non-zero complex constants, $|\mu| > \sup_z |e^{-\sigma z}|$.

It is natural to ask the following question about the composition of harmonic mappings with harmonic mappings.

Question 1.1. What is the harmonic mapping if all its pre-compositions or post-compositions by any harmonic mapping are still harmonic?

The first aim of this paper is to discuss Question 1.1 and similar questions about the composition of harmonic mappings with analytic functions, and the composition of biharmonic mappings with analytic functions or harmonic mappings. The second aim of this paper is to discuss the Goodman-Saff conjecture for biharmonic mappings (see Section 2).

2 Main Results

Our results are as follows.

Theorem 2.1. *Let f be a harmonic mapping. Then*

1. *for any harmonic F, $F \circ f$ is harmonic if and only if f is analytic or anti-analytic.*

2. *for any harmonic F, $f \circ F$ is harmonic if and only if $f(z) = az + b\overline{z} + c$, where a, b and c are constants.*

3. *for any harmonic F, both $f \circ F$ and $F \circ f$ are harmonic if and only if $f(z) = az + c$ or $f(z) = b\overline{z} + c$, where a, b and c are constants.*

4. *for any analytic F, $F \circ f$ is biharmonic if and only if f is analytic or anti-analytic.*

5. *for any harmonic F which is not analytic, $F \circ f$ is biharmonic if and only if f is analytic or anti-analytic.*

6. *for any biharmonic F which is not harmonic, $F \circ f$ is biharmonic if and only if $f(z) = az + c$ or $f(z) = b\overline{z} + c$, where a, b and c are constants.*

It is worth recalling that if f is an affine mapping, then for any harmonic mapping F, the composition $f \circ F$ is still harmonic. This fact is easy to verify, however.

Theorem 2.2. *Let f be an analytic function. Then*

1. *$f(z) = az + b$, where a and b are constants, if there exists some non-constant harmonic mapping F which is neither analytic nor anti-analytic such that $F \circ f$ is harmonic.*

2. *for any biharmonic F, $f \circ F$ or $F \circ f$ is harmonic if and only if $f(z) = c$, where c is a constant.*

3. *for any harmonic F, $f \circ F$ is biharmonic if and only if $f(z) = az + b$, where a and b are constants.*

4. *for any biharmonic F, $F \circ f$ is biharmonic if and only if $f(z) = az + b$, where a and b are constants.*

5. *for any biharmonic F which is not harmonic, $f \circ F$ is biharmonic if and only if $f(z) = az + b$, where a and b are constants.*

The next two theorems deal with cases that are not covered in Theorems 2.1 and 2.2.

Theorem 2.3. *Let f be a non-constant biharmonic mapping in a simply connected domain $D \subset \mathbb{C}$. Then for any analytic function F in D, $f \circ F$ is biharmonic if and only if f is harmonic.*

Theorem 2.4. *Let f be a harmonic mapping in a simply connected domain $D \subset \mathbb{C}$. Let F be either a* harmonic *function in D or a* biharmonic *function in D which is* not *harmonic. Then $f \circ F$ is biharmonic if and only if $f(z) = az + b\overline{z} + c$, where a, b and c are constants.*

To state our final result, we need some preparation. It is well-known that if an analytic function maps the unit disk $\mathbb{D}$ univalently onto a convex domain, then it also maps each concentric subdisk onto a convex domain (cf. [4]). It is natural to ask to what extent the special properties of conformal mappings will generalize to harmonic mappings of the disk onto convex domains. Goodman and Saff ([6]) constructed an example of a function convex in the vertical direction whose restriction to the disk $|z| < r$ does not have that property for any radius r in the interval $\sqrt{2} - 1 < r < 1$. In the same paper, they conjectured that the radius $\sqrt{2} - 1$ is best possible.

Definition 2.5. A domain Ω is convex in the direction $e^{i\phi}$, if for every fixed complex number a, the set $\Omega \cap \{a + te^{i\phi} : t \in \mathbb{R}\}$ is either connected or empty.

Let $\mathcal{K}(\phi)$ ($\mathcal{K}_H(\phi)$ resp.) denote the class of all complex-valued analytic (harmonic resp.) univalent functions f on the unit disk $\mathbb{D}$ with $f(\mathbb{D})$ convex in the direction $e^{i\phi}$. If $f \in \mathcal{K}(\phi)$ ($\mathcal{K}_H(\phi)$ resp.) is such that $f(\mathbb{D})$ *is convex in every direction* (i.e. $f(\mathbb{D})$ *is a convex domain*), then in this case we say that $f \in \mathcal{K}$ ($\mathcal{K}_H$ resp.).

Ruscheweyh and Salinas [11, Theorem 1] ultimately succeeded in proving the Goodman-Saff conjecture by showing that if $f \in \mathcal{K}_H(\phi)$, $0 < r \leq r_0 = \sqrt{2} - 1$, then one has $f(rz) \in \mathcal{K}_H(\phi)$. In particular, this gives

Theorem C. *Let* $f \in \mathcal{K}_H$, $0 < r \leq r_0 = \sqrt{2} - 1$. *Then* $f(rz) \in \mathcal{K}_H$.

In view of the development in the class of biharmonic mappings, it is interesting to ask whether the same conjecture holds for biharmonic mappings. That is, to what extent the special properties of conformal mappings will generalize to biharmonic mappings on the disk onto convex domains.

We now introduce the following notations:

$$\mathcal{H} = \{f \colon f \text{ is harmonic univalent in } \mathbb{D} \text{ with } f(0) = 0\}$$

and

$$\mathcal{BH} = \{g \colon g = \lambda_1 |z|^2 f + \lambda_2 f, \text{ where } f \in \mathcal{H}, \lambda_1 \text{ and } \lambda_2 \ (\lambda_1^2 + \lambda_2^2 \neq 0) \text{ are constants}\}.$$

Obviously, we have the following.

Proposition 2.6. *For any* $F \in \mathcal{BH}$, *if* $F = \lambda_1 |z|^2 f + \lambda_2 f$, *where* $f \in \mathcal{H}$, *then* F *is harmonic if and only if* $\lambda_1 = 0$.

For the second aim of this paper, we consider the conjecture mentioned above and obtain the following result.

Theorem 2.7. *For any non-constant* $F \in \mathcal{BH}$, F *sends the subdisk* $|z| < r$ *onto a convex region for* $r \leq \sqrt{2} - 1$, *but onto a non-convex region for any* $\sqrt{2} - 1 < r < 1$.

Since every harmonic mapping is biharmonic, Theorem 2.7 implies Theorem C.

We will prove Theorems 2.1 $-$2.4 in Section 3, where Theorem 2.1 is a solution to Question 1.1. In Section 4, we will prove Theorem 2.7.

3 The proofs of Theorems 2.1 $-$2.4

For any two C^1-functions f and g for which $f \circ g$ is defined, it is easy to see that the following chain rule holds, where $z = g(\zeta)$:

- $(f \circ g)_\zeta = \dfrac{\partial}{\partial \zeta}(f \circ g) = f_z g_\zeta + f_{\overline{z}} \overline{g_{\overline{\zeta}}}$
- $(f \circ g)_{\overline{\zeta}} = \dfrac{\partial}{\partial \overline{\zeta}}(f \circ g) = f_z g_{\overline{\zeta}} + f_{\overline{z}} \overline{g_\zeta}$
- $\overline{\left(\dfrac{\partial f}{\partial z}\right)} = \dfrac{\partial \overline{f}}{\partial \overline{z}}$.

3.1. Proof of Theorem 2.1. The sufficiency parts of the statements $(1) - (6)$ are obvious. Hence we only need to prove the corresponding necessary parts.

Necessary part in (1)**.** Let $H(z) = (F \circ f)(z)$ be harmonic, where F is harmonic and $\xi = f(z)$. By the chain rule, we have

$$H_z = F_\xi f_z + F_{\overline{\xi}} \overline{f_{\overline{z}}}$$

and, because f is assumed to be harmonic, we have

$$H_{z\overline{z}} := \frac{\partial^2 H}{\partial z \partial \overline{z}} = F_{\xi\xi} f_{\overline{z}} f_z + F_{\overline{\xi}\,\overline{\xi}}\, \overline{f_z}\, \overline{f_{\overline{z}}}. \tag{3.2}$$

Let $F(z) = z + \frac{1}{2}\overline{z}^2$. Then, from (3.2) and $H_{z\overline{z}} = 0$, we see that

$$\overline{f_{\overline{z}}}\, \overline{f_z} = 0.$$

It follows that $f_{\overline{z}} = 0$ or $\overline{f_z} = 0$. Hence, f is either analytic or anti-analytic.

Necessary part in (2)**.** Let $H(z) = (f \circ F)(z)$ be harmonic, where f is harmonic and $\zeta = F(z)$. By the chain rule, we have

$$H_z = f_\zeta F_z + f_{\overline{\zeta}} \overline{F_{\overline{z}}}$$

and, because f is harmonic, we easily have

$$H_{z\overline{z}} = f_{\zeta\zeta} F_{\overline{z}} F_z + f_{\overline{\zeta}\,\overline{\zeta}} \overline{F_z}\, \overline{F_{\overline{z}}}. \tag{3.3}$$

Let $F(z) = az + b\overline{z}$, where a and b are constants. Then from (3.3) and $H_{z\overline{z}} = 0$, it follows that

$$ab f_{\zeta\zeta} + \overline{a}\,\overline{b} f_{\overline{\zeta}\,\overline{\zeta}} = 0. \tag{3.4}$$

Setting $ab = 1$ first and then $ab = i$, we obtain from (3.4) that $f_{\zeta\zeta} = f_{\overline{\zeta}\,\overline{\zeta}} = 0$. Hence, f has the desired form

$$f(\zeta) = a\zeta + b\overline{\zeta} + c.$$

Necessary part in (3)**.** The conclusion follows from Parts (1) and (2).

Necessary part in (4)**.** Choose $F(z) = z^2$ and then $F(z) = z^3$. The biharmonicity of H leads to

$$f_{zz}f_{\overline{z}\,\overline{z}} = 0 \text{ and } f f_{zz}f_{\overline{z}\,\overline{z}} + (f_{\overline{z}})^2 f_{zz} + (f_z)^2 f_{\overline{z}\,\overline{z}} = 0. \tag{3.5}$$

Equation (3.5) yields that $f_z = 0$ or $f_{\overline{z}} = 0$ or $f(z) = az + b\overline{z} + c$, where a, b and c are constants. By letting $F(z) = z^4$, we see that $a = 0$ or $b = 0$.

Necessary part in (5)**.** Setting $F(z) = \overline{z}^2$ and $F(z) = \overline{z}^3$, we obtain that

$$\overline{f_{zz}}\,\overline{f_{\overline{z}\,\overline{z}}} = 0 \text{ and } \overline{f}\,\overline{f_{zz}}\,\overline{f_{\overline{z}\,\overline{z}}} + (\overline{f_{\overline{z}}})^2\overline{f_{zz}} + (\overline{f_z})^2\,\overline{f_{\overline{z}\,\overline{z}}} = 0. \tag{3.6}$$

Equation (3.6) yields that

$$f_z = 0 \text{ or } f_{\overline{z}} = 0 \text{ or } f(z) = az + b\overline{z} + c,$$

where a, b and c are constants. By letting $F(z) = \overline{z}^4$ we see that $a = 0$ or $b = 0$.

Necessary part in (6)**.** Let $F(z) = z\overline{z}$ and $H = F \circ f$, where f is harmonic. Then $H(z) = f(z)\overline{f(z)}$ and we have that

$$H_z = f_z\overline{f} + f\overline{f_{\overline{z}}}$$

and so,

$$h(z) = H_{z\overline{z}} = f_z\overline{f_z} + f_{\overline{z}}\overline{f_{\overline{z}}}.$$

This gives that

$$h_z = f_{zz}\overline{f_z} + f_{\overline{z}}\overline{f_{\overline{z}\,\overline{z}}}$$

and therefore, the biharmonicity of H is equivalent to

$$h_{z\overline{z}} = f_{zz}\overline{f_{zz}} + f_{\overline{z}\,\overline{z}}\,\overline{f_{\overline{z}\,\overline{z}}} = |f_{zz}|^2 + |f_{\overline{z}\,\overline{z}}|^2 = 0$$

which gives that

$$f_{zz} = f_{\overline{z}\,\overline{z}} = 0. \tag{3.7}$$

Equation (3.7) yields that

$$f(z) = \alpha z + \beta\overline{z} + \gamma,$$

where α, β, and γ are constants.

Let $F(z) = |z|^2 + z^4$. In this case, the condition $h_{z\overline{z}} = 0$ gives that $\alpha\beta = 0$. The proof is complete. ∎

3.8. Proof of Theorem 2.2. Again, as the sufficiencies in statements (1) − (5) are obvious, we only need to prove their necessities.

Necessary part in (1)**.** Let $H(z) = f(F(z))$ and $\zeta = F(z)$. Then

$$H_{z\overline{z}} = f''\frac{\partial\zeta}{\partial z}\frac{\partial\zeta}{\partial\overline{z}} = 0$$

which gives $f''(\zeta)=0$ and therefore $f(\zeta)=a\zeta+b$, where a and b are constants.

Necessary part in (2)**.** Let $H(z)=(f\circ F)(z)=f(\zeta)$ be harmonic, where $\zeta=F(z)=z\overline{z}$. Then $H_z=\overline{z}(f'\circ F)$ and

$$H_{z\overline{z}}=\overline{z}((f''\circ F)z)+(f'\circ F)=|z|^2(f''\circ F)+f'\circ F=0$$

so that $\zeta f''(\zeta)+f'(\zeta)=0$. This gives

$$f(\zeta)=c_0\log\zeta+c_1,$$

where c_0 and c_1 are constants.

Let $F(z)=(z+\overline{z})|z|^2$. Then we have that $c_0=0$.

Next we assume that $f\circ F$ is harmonic, where $F(z)=z\overline{z}$. Then $F\circ f=f\overline{f}$ and

$$(F\circ f)_{z\overline{z}}=|f'|^2=0.$$

Hence, $f(z)$ must be a constant.

Necessary part in (3)**.** Let $\zeta=F(z)=z+\overline{z}$. Then

$$(f\circ F)_{\zeta\overline{\zeta}\zeta\overline{\zeta}}=f^{(4)}=0$$

which shows that $f(\zeta)$ reduces to a cubic polynomial in ζ:

$$f(\zeta)=a_1\zeta^3+a_2\zeta^2+a\zeta+b.$$

However, if we let $F(z)=\frac{1}{2}(z^2+\overline{z}^2)$, then it follows that $a_1=a_2=0$. Thus, $f(\zeta)$ becomes a linear function.

Necessary part in (4)**.** Let $H(z)=(F\circ f)(z)$ and $F(z)=|z|^2$. Then $H=f\overline{f}$ and so,

$$H_{z\overline{z}}=|f'|^2\ \text{ and }\ H_{z\overline{z}z\overline{z}}=|f''|^2=0.$$

This gives that $f''(z)=0$ and therefore $f(z)=az+b$, where a and b are constants.

Necessary part in (5)**.** Obviously, $f(z)=az+b$ satisfies the requirements. Suppose there exists some analytic function f satisfying the requirements but is not in the form $az+b$.

Let $H(\zeta)=f(F(\zeta))=f(z)$, where $z=F(\zeta)$. Then

$$\frac{\partial^2 H}{\partial\zeta\partial\overline{\zeta}}=f''\frac{\partial z}{\partial\overline{\zeta}}\frac{\partial z}{\partial\zeta}+f'\frac{\partial^2 z}{\partial\overline{\zeta}\partial\zeta}$$

and

$$\begin{aligned}\frac{\partial^4 H}{\partial\zeta\partial\overline{\zeta}\partial\zeta\partial\overline{\zeta}} &= f^{(4)}\Big(\frac{\partial z}{\partial\zeta}\Big)^2\Big(\frac{\partial z}{\partial\overline{\zeta}}\Big)^2\\ &+ f'''\left[4\frac{\partial z}{\partial\overline{\zeta}}\frac{\partial z}{\partial\zeta}\frac{\partial^2 z}{\partial\zeta\partial\overline{\zeta}}+\frac{\partial^2 z}{\partial\overline{\zeta}^2}\Big(\frac{\partial z}{\partial\zeta}\Big)^2+\frac{\partial^2 z}{\partial\zeta^2}\Big(\frac{\partial z}{\partial\overline{\zeta}}\Big)^2\right]\\ &+ f''\left[2\frac{\partial^3 z}{\partial\zeta\partial^2\overline{\zeta}}\frac{\partial z}{\partial\zeta}+2\Big(\frac{\partial^2 z}{\partial\zeta\partial\overline{\zeta}}\Big)^2+2\frac{\partial^3 z}{\partial\overline{\zeta}\partial^2\zeta}\frac{\partial z}{\partial\overline{\zeta}}+\frac{\partial^2 z}{\partial\zeta^2}\frac{\partial^2 z}{\partial\overline{\zeta}^2}\right].\end{aligned}$$

Let $z = F(\zeta) = \zeta\overline{\zeta}$. Then we have that

$$\frac{\partial^4 H}{\partial\zeta\partial\overline{\zeta}\partial\zeta\partial\overline{\zeta}} = |\zeta|^4 f^{(4)} + 4|\zeta|^2 f''' + 2f'' = 0$$

and, as $z = |\zeta|^2$, this gives

$$z^2 f^{(4)} + 4zf''' + 2f'' = 0. \tag{3.9}$$

Solving this equation for f'' gives

$$f''(z) = \frac{c_0}{z} + \frac{c_1}{z^2},$$

where c_0 and c_1 are constants, if the domain D of $f(z)$ does not contain the origin O. If D contains O, then the only solution to (3.9) is given by $f''(z) = 0$ which gives $f(z) = az + b$, a contradiction.

If D does not contain the origin O, then the last equation gives the solution

$$f(z) = c_0 z(\log z - 1) - c_1 \log z + c_2 z + c_3,$$

where c_2 and c_3 are constants.

But, an elementary computation shows that $c_0 = c_1 = 0$ and hence, $f(z)$ reduces to linear functions. This is again a contradiction and we finish the proof. ■

3.10. Proof of Theorem 2.3. The sufficiency is obvious and therefore, we need to prove the necessary part of the theorem. Since the domain $D \subset \mathbb{C}$ is simply connected, by [1, 2], we may assume that f has the representation $f = |z|^2 G + K$, where G and K are harmonic in D. Then the proof will be completed once we show that $G(z) \equiv 0$.

Suppose not. Since a harmonic function of an analytic function is known to be harmonic, it suffices to consider the case

$$f(z) = |z|^2 G(z).$$

Now, we let

$$H(z) = f(F(z)) = F(z)\overline{F(z)}G(F(z))$$

and $P(z) = G(F(z)) = f_1(z) + \overline{f_2(z)}$, where f_1 and f_2 are analytic. Then

$$h = H_{z\overline{z}} = F'\overline{F'}(f_1 + f_2) + F'\overline{F}\,\overline{f_2} + Ff_1'\overline{F'}$$

and

$$\begin{aligned} h_{z\overline{z}} = H_{z\overline{z}z\overline{z}} &= F''\overline{F''}f_1 + F''\overline{F''}\,\overline{f_2} + F''\overline{F'}\,\overline{f_2'} + \overline{F''}F'f_1' \\ &\quad + F''\overline{F'}\,\overline{f_2'} + F''\overline{F}\,\overline{f_2''} + F'\overline{F''}f_1' + f_1''\overline{F''}F \\ &= \overline{F''}\left(F''f_1 + 2F'f_1' + Ff_1''\right) \\ &\quad + F''\left(\overline{F''}\,\overline{f_2} + 2\overline{F'}\,\overline{f_2'} + \overline{F}\,\overline{f_2''}\right). \end{aligned}$$

The hypothesis $\Delta(\Delta H) \equiv 0$ implies that

$$\overline{F''}\left(F''f_1 + 2F'f_1' + Ff_1''\right) = -F''\left(\overline{F''}\,\overline{f_2} + 2\overline{F'}\,\overline{f_2'} + \overline{F}\,\overline{f_2''}\right). \tag{3.11}$$

Let $F(z) = z^2$. Then (3.11) yields that

$$2f_1 + 2zf_1' + z^2f_1'' = -(\overline{2f_2 + 2zf_2' + z^2f_2''}).$$

The fact that a function which is analytic and anti-analytic simultaneously must be constant shows that there is a constant c satisfying the following:

$$2f_1 + 2zf_1' + z^2f_1'' = 2c \tag{3.12}$$

and

$$2f_2 + 2zf_2' + z^2f_2'' = -2\overline{c}. \tag{3.13}$$

It follows from the equations (3.12) and (3.13) that

$$f_1(z) = c_1z^{\lambda_1} + c_2z^{\lambda_2} + c \tag{3.14}$$

and

$$f_2(z) = c_3z^{\lambda_1} + c_4z^{\lambda_2} - \overline{c}, \tag{3.15}$$

where c_1, c_2, c_3, c_4 are constants, $\lambda_1 = \frac{-1+i\sqrt{7}}{2}$, and $\lambda_2 = \overline{\lambda_1}$. Hence

$$G \circ F(z) = f_1(z) + \overline{f_2(z)} = G(z^2). \tag{3.16}$$

We first consider the case when the origin O is not contained in D.

Let $F(z) = z^4$. From (3.11), we know that

$$f_1(z) = c_5z^{-3} + c_6z^{-4} + c_0 \tag{3.17}$$

and

$$f_2(z) = c_7z^{-3} + c_8z^{-4} - \overline{c_0}, \tag{3.18}$$

where c_0, c_5, c_6, c_7, c_8 are constants. Then

$$G(F(z)) = f_1(z) + \overline{f_2(z)} = G(z^4). \tag{3.19}$$

It follows from (3.14) $-$ (3.19) that $G \equiv 0$. Also (3.17) and (3.18) imply that if O is contained in D, then $G \equiv 0$. This is the desired contradiction. The proof of Theorem 2.3 is complete. ∎

3.20. Proof of Theorem 2.4. The sufficiency is obvious. Therefore, we need to prove only the necessary part of the theorem.

Let f be harmonic in D. Then, f has the representation $f = h + \overline{g}$, where h and g are analytic in D.

Let $H(\zeta) = (f \circ F)(\zeta) = f(z)$, where $z = F(\zeta)$ and F is *harmonic*. Then we have

$$H_{\zeta\overline{\zeta}} = h''F_\zeta F_{\overline{\zeta}} + \overline{g''}\,\overline{F_\zeta}\,\overline{F_{\overline{\zeta}}}$$

and

$$\begin{aligned} H_{\zeta\overline{\zeta}\zeta\overline{\zeta}} &= h^{(4)}(F_{\zeta})^2(F_{\overline{\zeta}})^2 + h'''\left((F_{\zeta})^2F_{\overline{\zeta}\overline{\zeta}} + (F_{\overline{\zeta}})^2F_{\zeta\zeta}\right) + h''F_{\overline{\zeta}\,\overline{\zeta}}F_{\zeta\zeta} \\ &\quad + \overline{g^{(4)}}(\overline{F_{\zeta}})^2(\overline{F_{\overline{\zeta}}})^2 + \overline{g'''}\left((\overline{F_{\overline{\zeta}}})^2\overline{F_{\zeta\zeta}} + (\overline{F_{\zeta}})^2\overline{F_{\overline{\zeta}\,\overline{\zeta}}}\right) + \overline{g''}\,\overline{F_{\overline{\zeta}\,\overline{\zeta}}}\,\overline{F_{\zeta\zeta}}. \end{aligned}$$

Now, we let $z = F(\zeta) = \zeta + \overline{\zeta}$. Then the last equation reduces to

$$H_{\zeta\overline{\zeta}\zeta\overline{\zeta}} = h^{(4)} + \overline{g^{(4)}} = 0,$$

which yields that

$$f(z) = a_3z^3 + a_2z^2 + az + \overline{b}_3\overline{z}^3 + \overline{b}_2\overline{z}^2 + b\overline{z} + c.$$

Moreover, if we let $F(z) = z^2 + \overline{z}^3$ then we have $a_3 = a_2 = b_3 = b_2 = 0$ and therefore, f has the form $f(z) = az + b\overline{z} + c$.

Next, we consider the case $H(\zeta) = (f \circ F)(\zeta) = f(z)$, where $z = F(\zeta)$ and F is *biharmonic*. In this case, we see that

$$H_{\zeta\overline{\zeta}} = h''F_{\zeta}F_{\overline{\zeta}} + h'F_{\zeta\overline{\zeta}} + \overline{g}''\overline{F}_{\zeta}\overline{F}_{\overline{\zeta}} + \overline{g}'\overline{F}_{\zeta\overline{\zeta}}$$

and

$$\begin{aligned} H_{\zeta\overline{\zeta}\zeta\overline{\zeta}} &= h^{(4)}(F_{\zeta})^2(F_{\overline{\zeta}})^2 + h'''(4F_{\zeta}F_{\overline{\zeta}}F_{\zeta\overline{\zeta}} + (F_{\zeta})^2F_{\overline{\zeta}\,\overline{\zeta}} + (F_{\overline{\zeta}})^2F_{\zeta\zeta}) \\ &\quad + h''(F_{\overline{\zeta}\overline{\zeta}}F_{\zeta\zeta} + 2F_{\zeta\zeta\overline{\zeta}}F_{\overline{\zeta}} + 2F_{\zeta\overline{\zeta}}^2 + 2F_{\zeta}F_{\zeta\overline{\zeta}\,\overline{\zeta}}) + h'F_{\zeta\overline{\zeta}\zeta\overline{\zeta}} \\ &\quad + \overline{g^{(4)}}(\overline{F_{\zeta}})^2(\overline{F_{\overline{\zeta}}})^2 + \overline{g'''}(4\overline{F_{\zeta}}\,\overline{F_{\overline{\zeta}}}\,\overline{F_{\zeta\overline{\zeta}}} + (\overline{F_{\overline{\zeta}}})^2\overline{F_{\zeta\zeta}} + (\overline{F_{\zeta}})^2\overline{F_{\overline{\zeta}\,\overline{\zeta}}}) \\ &\quad + \overline{g''}(\overline{F_{\overline{\zeta}\,\overline{\zeta}}}\,\overline{F_{\zeta\zeta}} + 2\overline{F_{\zeta\zeta\overline{\zeta}}}\,\overline{F_{\overline{\zeta}}} + 2\overline{F_{\zeta\overline{\zeta}}}^2 + 2\overline{F_{\zeta}}\,\overline{F_{\zeta\overline{\zeta}\,\overline{\zeta}}}) + \overline{g'}\overline{F_{\zeta\overline{\zeta}\zeta\overline{\zeta}}}. \end{aligned}$$

Now, we make the choice $z = F(\zeta) = t\zeta\overline{\zeta}$, where t is a real constant. Then

$$H_{\zeta\overline{\zeta}\zeta\overline{\zeta}} = t^4h^{(4)} + 4zt^3h''' + 2t^2h'' + t^4\overline{g}^{(4)} + 4\overline{z}t^3\overline{g}''' + 2t^2\overline{g}'' = 0. \tag{3.21}$$

From (3.21), we have that

$$t^4h^{(4)} + 4zt^3h''' + 2t^2h'' = c_0 \tag{3.22}$$

and

$$t^4g^{(4)} + 4zt^3g''' + 2t^2g'' = -\overline{c_0}, \tag{3.23}$$

where c_0 is a constant.

Equations (3.22) and (3.23) show that $g'' = h'' = 0$ and hence, f takes the form

$$f(z) = az + b\overline{z} + c,$$

where a, b and c are constants. The proof is finished. ■

4 Proof of Theorem 2.7

The following lemma is crucial for the proof of Theorem 2.7.

Lemma 4.1. *For any $F \in \mathcal{BH}$,*

$$\frac{\partial}{\partial t}\Big(\arg\frac{\partial F(re^{it})}{\partial t}\Big) = \frac{\partial}{\partial t}\Big(\arg\frac{\partial f(re^{it})}{\partial t}\Big).$$

Proof. Let $F \in \mathcal{BH}$ and $z = re^{it}$. Then, by the definition, F has the form

$$F(z) = \lambda_1|z|^2 f + \lambda_2 f = \lambda_1 r^2 f + \lambda_2 f$$

where $f \in \mathcal{H}$, λ_1 and λ_2 are constants with $\lambda_1^2 + \lambda_2^2 \neq 0$. As $\lambda_1 r^2 + \lambda_2$ is independent of t, we easily have

$$\frac{\partial F(re^{it})}{\partial t} = (\lambda_1 r^2 + \lambda_2)\frac{\partial f(re^{it})}{\partial t}$$

and

$$\frac{\partial^2 F(re^{it})}{\partial t^2} = (\lambda_1 r^2 + \lambda_2)\frac{\partial^2 f(re^{it})}{\partial t^2}.$$

Hence

$$\frac{\partial}{\partial t}\Big(\arg\frac{\partial F(re^{it})}{\partial t}\Big) = \mathrm{Im}\left(\frac{\frac{\partial^2 F(re^{it})}{\partial t^2}}{\frac{\partial F(re^{it})}{\partial t}}\right) = \mathrm{Im}\left(\frac{\frac{\partial^2 f(re^{it})}{\partial t^2}}{\frac{\partial f(re^{it})}{\partial t}}\right) = \frac{\partial}{\partial t}\Big(\arg\frac{\partial f(re^{it})}{\partial t}\Big)$$

and the proof of the lemma is complete. ∎

4.2. Proof of Theorem 2.7. The proof follows from Lemma 4.1 and Theorem C. ∎

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