Optimal Strategies for Symmetric Matrix Games with Partitions

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Abstract

We introduce three variants of a symmetric matrix game corresponding to three ways of comparing two partitions of a fixed integer (σ) into a fixed number (n) of parts. In the random variable interpretation of the game, each variant depends on the choice of a copula that binds the marginal uniform cumulative distribution functions (cdf) into the bivariate cdf. The three copulas considered are the product copula $T_{\mathbf{P}}$ and the two extreme copulas, i.e. the minimum copula $T_{\mathbf{M}}$ and the Łukasiewicz copula $T_{\mathbf{L}}$. The associated games are denoted as the $(n,\sigma)_{\mathbf{P}}$, $(n,\sigma)_{\mathbf{M}}$ and $(n,\sigma)_{\mathbf{L}}$ games. In the present paper, we characterize the optimal strategies of the $(n,\sigma)_{\mathbf{M}}$ and $(n,\sigma)_{\mathbf{L}}$ games and compare them to the optimal strategies of the $(n,\sigma)_{\mathbf{P}}$ games. It turns out that the characterization of the optimal strategies is completely different for each game variant.

1 Description of the games

1.1 Preliminary concepts

Consider a collection $\{X_1, X_2, \ldots, X_m\}$ of discrete random variables that are uniformly distributed on integer multisets. Any two random variables X_i and X_j can be statistically compared, yielding the probabilistic relation $Q = [q_{ij}]$ generated by the collection, defined by

$$q_{ij} = \text{Prob}\{X_i > X_j\} + \frac{1}{2}\text{Prob}\{X_i = X_j\}. \tag{1}$$

Note that q_{ij} denotes the winning probability of X_i w.r.t. X_j. The term probabilistic relation refers to the property that $q_{ij} + q_{ji} = 1$, for any i and j, in particular $q_{ii} = 1/2$ for any i. This probabilistic relation, in particular its transitivity [2], has been studied in various settings (see, e.g., [3, 4, 6, 7]).

In the present paper, the random variables are uniformly distributed on multisets of n strictly positive integers summing up to σ, where n and σ are given fixed numbers. Each element of the multiset therefore has the same probability $1/n$. In partition theory, an ordered multiset of n strictly positive integers summing up to σ is known as a partition of σ into n parts. Throughout this paper, we will maintain the latter terminology.

Definition 1.1. The n-tuple $\pi = (i_1, i_2, \ldots, i_n)$ consisting of n strictly positive integers ordered nondecreasingly and with collective sum equal to σ, is called a partition of σ into n parts. We will denote this type of partition by an (n, σ) partition.

The integers composing a partition are called the parts of that partition. Note that in partition theory, the parts are usually ordered nonincreasingly. Throughout this paper, when considering an (n, σ) partition π_1, resp. π_2, the parts will be denoted as $(i_1, i_2, \ldots, i_n)$, resp. $(i'_1, i'_2, \ldots, i'_n)$ (the primes distinguishing partition π_2 from π_1). It is sometimes helpful to use a notation that makes explicit the number of times a particular integer appears in a partition. We use the same notation as in partition theory, known as the multiplicity representation of the partition.

Definition 1.2. The multiplicity representation of an (n, σ) partition π is given by $(1^{t_1} 2^{t_2} 3^{t_3} \ldots)$ in which t_i denotes the number of times i appears in the partition π. When $t_i = 0$ the entry i^{t_i} can be omitted.

For the multiplicity representation $(1^{t_1} 2^{t_2} 3^{t_3} \ldots)$ of a given (n, σ) partition π it clearly holds that $0 \leq t_i \leq n$, $\sum_{i>0} t_i = n$ and $\sum_{i>0} i t_i = \sigma$.

In the next subsection, we will define three variants of the same game. The payoff matrix of this game, needed for determining the corresponding optimal strategies, is completely determined by the probabilistic relation generated by the collection of random variables. This probabilistic relation depends upon the copula used for coupling the random variables.

It is well known that for discrete random variables X_i and X_j the probability $p_{X_i, X_j}(k, l)$ that X_i takes value k and X_j takes value l, can be obtained from the joint cumulative distribution function F_{X_i, X_j} as follows:

$$p_{X_i, X_j}(k, l) = F_{X_i, X_j}(k, l) + F_{X_i, X_j}(k-1, l-1) - F_{X_i, X_j}(k, l-1) - F_{X_i, X_j}(k-1, l).$$

Sklar's theorem [10] says that if a joint cumulative distribution function F_{X_i, X_j} has marginals F_{X_i} and F_{X_j}, then there exists a copula C such that for all x, y:

$$F_{X_i, X_j}(x, y) = C(F_{X_i}(x), F_{X_j}(y)). \tag{2}$$

On the other hand, if C is a copula and F_{X_i} and F_{X_j} are cumulative distribution functions, then the function defined by (2) is a joint cumulative distribution function with marginals F_{X_i} and F_{X_j}. Let us recall [8, 9] that a copula is a binary

operation $C : [0,1]^2 \to [0,1]$ that has neutral element 1 and absorbing element 0 and that satisfies the property of moderate growth: for any $(x_1, x_2, y_1, y_2) \in [0,1]^4$

$$(x_1 \leq x_2 \wedge y_1 \leq y_2) \Rightarrow C(x_1, y_1) + C(x_2, y_2) \geq C(x_1, y_2) + C(x_2, y_1).$$

All copulas are situated between the Łukasiewicz copula $T_{\mathbf{L}}(x,y) = \max(0, x + y - 1)$ (also called Fréchet-Hoeffding lower bound and denoted W) and the minimum copula $T_{\mathbf{M}}(x,y) = \min(x,y)$ (also called Fréchet-Hoeffding upper bound and denoted M).

1.2 The three game variants

We consider three variants of the same game, played between two players who want to maximize their individual profit. The game is therefore a non-cooperative game. The strategies for both players are the (n, σ) partitions, with n and σ fixed before the game begins. As already mentioned, with each (n, σ) partition π_i we let correspond a random variable X_i that is uniformly distributed on the partition parts. The payoff matrix for player 1 is then given by $A = [a_{ij}]$, where $a_{ij} = q_{ij} - 1/2$ and q_{ij} is given by (1). As $Q = [q_{ij}]$ is a probabilistic relation, it holds that $a_{ij} = -a_{ji}$ and the game therefore is a symmetric matrix game. For such games, it holds that the payoff in a saddle point is zero and it is well known that the optimal strategies of a matrix game are the strategies occurring in a saddle point. When verifying that some strategy π_i is optimal in the games discussed in this paper, it therefore suffices to verify that $q_{ij} \geq 1/2$, for all strategies π_j.

The three game variants differ from each other in the use of a different copula to couple pairwisely the random variables. The connection to copulas will be shown in the next subsection. For two (n, σ) partitions $\pi_i = (i_1, \ldots, i_n)$ and $\pi_j = (j_1, \ldots, j_n)$:

(i) the first game variant defines q_{ij} as

$$q_{ij}^{\mathbf{P}} = \frac{\#\{(k,l) \mid i_k > j_l\}}{n^2} + \frac{\#\{(k,l) \mid i_k = j_l\}}{2n^2}, \tag{3}$$

(ii) the second game variant defines q_{ij} as

$$q_{ij}^{\mathbf{M}} = \frac{\#\{k \mid i_k > j_k\}}{n} + \frac{\#\{k \mid i_k = j_k\}}{2n}, \tag{4}$$

(iii) and the third game variant defines q_{ij} as

$$q_{ij}^{\mathbf{L}} = \frac{\#\{k \mid i_k > j_{n-k+1}\}}{n} + \frac{\#\{k \mid i_k = j_{n-k+1}\}}{2n}. \tag{5}$$

One can verify that $Q = [q_{ij}]$ is, in all three game variants, a probabilistic relation. We say that an (n, σ) partition π_i wins, resp. loses, from an (n, σ) partition π_j if $q_{ij} > 1/2$, resp. $q_{ij} < 1/2$. When it is better suited to explicitly mention the partitions defining q_{ij}, we will use the notation Q_{π_i, π_j}.

The first (second, third) game variant is denoted as an $(n, \sigma)_\mathbf{P}$ game ($(n, \sigma)_\mathbf{M}$ game, $(n, \sigma)_\mathbf{L}$ game). Here, $\mathbf{P}$ refers to the product copula $T_\mathbf{P}$, $\mathbf{M}$ to the minimum copula $T_\mathbf{M}$ and $\mathbf{L}$ to the Łukasiewicz copula $T_\mathbf{L}$, which are the respective copulas used for coupling the random variables [9].

Consider e.g. the $(4, 16)$ partitions $\pi_1 = (1, 2, 5, 8)$ and $\pi_2 = (2, 3, 5, 6)$. Figure 1 shows graphically, for each considered game variant, which parts of the partitions have to be compared. We obtain $q_{12}^\mathbf{P} = (0 + 0.5 + 2.5 + 4)/16 = 7/16$, $q_{12}^\mathbf{M} = 0 + 0 + 1/8 + 1/4 = 3/8$ and $q_{12}^\mathbf{L} = 0 + 0 + 1/4 + 1/4 = 1/2$.
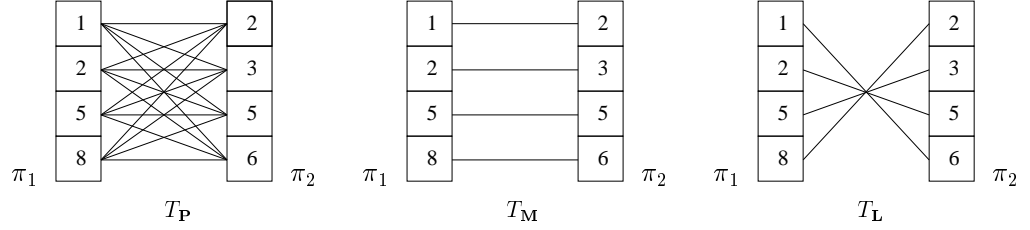

Figure 1: The three game types for a specific example.

In the last two sections of this paper, the optimal strategies of the $(n, \sigma)_\mathbf{M}$ and the $(n, \sigma)_\mathbf{L}$ games are laid bare. Both sections start with a subsection that bundles the results, after which a subsection follows in which these results are proven. For the sake of completeness, the optimal strategies of the $(n, \sigma)_\mathbf{P}$ game, which were already obtained by the present authors in [5], are presented in the next section. We end the present section with a proof of the connection between the probabilistic relation (4), resp. (5), and the copula $T_\mathbf{M}$, resp. $T_\mathbf{L}$.

1.3 Connection with the extreme copulas

For discrete random variables, equation (1), defining q_{ij}, can be restated as

$$q_{ij} = \sum_{k > l} p_{X_i, X_j}(k, l) + \frac{1}{2} \sum_{k = l} p_{X_i, X_j}(k, l). \qquad (6)$$

Assume now that the r.v. X_i, resp. X_j, with cumulative distribution function F_{X_i}, resp. F_{X_j}, and probability mass function p_{X_i}, resp. p_{X_j}, correspond to a multiset $(i_1, i_2, \ldots, i_n)$, resp. $(j_1, j_2, \ldots, j_n)$, the elements of the multisets ordered non-decreasingly and each element of the multiset having probability $1/n$. We first consider the probabilistic relation when using the copula $T_\mathbf{M}$. It then holds that

$$
\begin{aligned}
p_{X_i, X_j}^\mathbf{M}(k, l) \;=\; & \min(F_{X_i}(k), F_{X_j}(l)) + \min(F_{X_i}(k - 1), F_{X_j}(l - 1)) \\
& - \min(F_{X_i}(k), F_{X_j}(l - 1)) - \min(F_{X_i}(k - 1), F_{X_j}(l)),
\end{aligned}
$$

which is equivalent to:

$$
p_{X_i, X_j}^\mathbf{M}(k, l) = \begin{cases} 0 & , \text{ if } F_{X_i}(k) \le F_{X_j}(l - 1) \vee F_{X_j}(l) \le F_{X_i}(k - 1), \\ \min(F_{X_i}(k), F_{X_j}(l)) - \max(F_{X_i}(k - 1), F_{X_j}(l - 1)) \\ \qquad\qquad , \text{ otherwise}. \end{cases}
$$

As each element in the multiset has probability $1/n$, the first line of the above expression is equivalent to saying that when $\#\{\ell \mid i_\ell = k \wedge j_\ell = l\} = 0$, it holds that $p^{\mathbf{M}}_{X_i,X_j}(k,l) = 0$. The second line is then equivalent to saying that when $\#\{\ell \mid i_\ell = k \wedge j_\ell = l\} = f > 0$, it holds that $p^{\mathbf{M}}_{X_i,X_j}(k,l) = f/n$. Using (6), definition (4) now follows immediately.

Secondly, we consider the copula $T_{\mathbf{L}}$, and obtain:

$$p^{\mathbf{L}}_{X_i,X_j}(k,l) =$$
$$\max(F_{X_i}(k) + F_{X_j}(l) - 1, 0) + \max(F_{X_i}(k-1) + F_{X_j}(l-1) - 1, 0)$$
$$- \max(F_{X_i}(k) + F_{X_j}(l-1) - 1, 0) - \max(F_{X_i}(k-1) + F_{X_j}(l) - 1, 0),$$

which is equivalent to:

$$p^{\mathbf{L}}_{X_i,X_j}(k,l) = \begin{cases} 0 & , \text{if } F_{X_i}(k) \leq 1 - F_{X_j}(l) \vee 1 - F_{X_j}(l-1) \leq F_{X_i}(k-1), \\ \min(F_{X_i}(k), 1 - F_{X_j}(l-1)) - \max(F_{X_i}(k-1), 1 - F_{X_j}(l)) \\ & , \text{otherwise}. \end{cases}$$

The first line of the above expression is equivalent to demanding that when $\#\{\ell \mid i_\ell = k \wedge j_{n+\ell-1} = l\} = 0$, it holds that $p^{\mathbf{L}}_{X_i,X_j}(k,l) = 0$. The second part is then equivalent to saying that when $\#\{\ell \mid i_\ell = k \wedge j_{n+\ell-1} = l\} = f > 0$, it holds that $p^{\mathbf{L}}_{X_i,X_j}(k,l) = f/n$. Using (6), definition (5) follows immediately.

2   Optimal strategies for $(n,\sigma)_{\mathbf{P}}$ games

For the proofs of the statements in this section we refer to [5].

Theorem 2.1. *An $(n,\sigma)_{\mathbf{P}}$ game has at least one optimal strategy if and only if one of the following six mutually exclusive conditions is satisfied:*

(i) $n \leq 2$

(ii) $(n,\sigma) = (3,7)$

(iii) $(n,\sigma) = (3,8)$

(iv) $(n,\sigma) = (2l, 4l+1)$, $l > 1$

(v) $n > 2$ and there exist $a, b, k \in \mathbb{N}$ such that

$$\begin{cases} n = (a+b)k - b \\ \sigma = nk \end{cases} \tag{7}$$

(vi) $n > 2$ and there exist $a, b, k \in \mathbb{N}$ such that

$$\begin{cases} n = (a+b)k \\ \sigma = (n+b)k \\ a \neq 0 \wedge b \neq 0 \end{cases} \tag{8}$$

Proposition 2.2.

1. *The $(1, \sigma)_{\mathbf{P}}$ game: the unique strategy (σ) is optimal.*

2. *The $(2, \sigma)_{\mathbf{P}}$ game: all $\lfloor \frac{\sigma}{2} \rfloor$ different strategies are optimal.*

3. *The $(3, 7)_{\mathbf{P}}$ game: $(1^1 3^2)$ is the only optimal strategy.*

4. *The $(3, 8)_{\mathbf{P}}$ game: $(1^1 3^1 4^1)$ is the only optimal strategy.*

5. *The $(n, n)_{\mathbf{P}}$ game: the unique strategy (1^n) is optimal.*

6. *The $(2n, 4n + 1)_{\mathbf{P}}$ game, $n > 1$: $(1^{n-1} 2^1 3^n)$ is the only optimal strategy.*

Proposition 2.3. *All $(n, \sigma)_{\mathbf{P}}$ games, with $n \neq \sigma$, satisfying (7) have exactly $\lfloor a/(k-1) \rfloor + \lfloor b/k \rfloor + 1$ optimal strategies and their multiplicity representation is given by $(1^a 2^b 3^a 4^b \ldots (2k-2)^b (2k-1)^a)$, where a, b are different but k is the same for each optimal strategy.*

Proposition 2.4. *All $(n, \sigma)_{\mathbf{P}}$ games satisfying (8) have exactly one optimal strategy $(1^a 2^b 3^a 4^b \ldots (2k-1)^a (2k)^b)$.*

Two interesting corollaries follow from the above propositions.

Corollary 2.5. *For given values n and σ ($n \neq \sigma$), the entity $\lfloor a/(k-1) \rfloor + \lfloor b/k \rfloor$ is an invariant of the solution space of system (7). If this system has a solution, then it has exactly $\lfloor a/(k-1) \rfloor + \lfloor b/k \rfloor + 1$ solutions.*

Corollary 2.6. *For given values n and σ, the system (8) has at most one solution.*

We end this section with an example, namely the game that contains the classical dice usually encountered in games with dice.

Example 2.7. The $(6, 21)_{\mathbf{P}}$ game has 110 strategies and one optimal strategy, namely the classical dice $(1, 2, 3, 4, 5, 6)$, which is of type (8) with $a = b = 1$ and $k = 3$.

3   Optimal strategies for $(n, \sigma)_{\mathbf{M}}$ games

3.1   Results

The following lemma states a remarkable result about the integers occurring as parts of an optimal strategy in an $(n, \sigma)_{\mathbf{M}}$ game.

Lemma 3.1. *The only optimal strategy in an $(n, \sigma)_{\mathbf{M}}$ game, with $n \geq 3$, for which the highest part is strictly greater than 5 is $(2, 4, 6)$, a strategy of the $(3, 12)_{\mathbf{M}}$ game.*

The above lemma will be crucial in our proof of the following theorem.

Theorem 3.2. *An $(n, \sigma)_{\mathbf{M}}$ game has optimal strategies if and only if one of the following three mutually exclusive conditions is satisfied:*

(i) $n \leq 2$

(ii) $(n, \sigma) = (3, 12)$

(iii) $n > 2$ and there exist $t_1, \ldots, t_5 \in \mathbb{N}$ such that the following conditions are satisfied:

$$\begin{cases} t_1 + t_2 + t_3 + t_4 + t_5 = n \\ t_1 + 2t_2 + 3t_3 + 4t_4 + 5t_5 = \sigma \\ t_3 > 0 \Rightarrow t_2 + 2 > (t_3 - 1) + t_4 + t_5 \\ t_4 > 0 \Rightarrow t_3 + 2 > t_1 + (t_4 - 1) + t_5 \\ t_5 > 0 \Rightarrow t_4 + 2 > t_1 + t_2 + (t_5 - 1) \end{cases} \qquad (9)$$

We are also able to describe the optimal strategies of the $(n, \sigma)_{\mathbf{M}}$ games. We first handle the special cases.

Proposition 3.3.

1. The $(1, \sigma)_{\mathbf{M}}$ game: the unique strategy (σ) is optimal.

2. The $(2, \sigma)_{\mathbf{M}}$ game: all $\lfloor \frac{\sigma}{2} \rfloor$ different strategies are optimal.

3. The $(3, 12)_{\mathbf{M}}$ game: $(2, 4, 6)$ is the only optimal strategy.

All other optimal strategies are identified in the next proposition.

Proposition 3.4. *All optimal strategies of $(n, \sigma)_{\mathbf{M}}$ games that are not covered by Proposition 3.3 have a multiplicity representation $(1^{t_1} 2^{t_2} 3^{t_3} 4^{t_4} 5^{t_5})$, such that $(t_1, \ldots, t_5)$ is a solution of (9).*

However, a closed formula expressing the number of optimal strategies of an arbitrary $(n, \sigma)_{\mathbf{M}}$ game, has not yet been found.

Example 3.5. The $(5, 16)_{\mathbf{M}}$ game has 37 strategies and only one optimal strategy, namely $\pi = (2, 2, 3, 4, 5)$ for which $(t_1, t_2, t_3, t_4, t_5) = (0, 2, 1, 1, 1)$. One can easily verify that conditions (9) are satisfied for π. Moreover, none of the other $(5, 16)$ partitions satisfy these conditions.

3.2 Proof

We start this subsection by introducing increment and decrement operations, which will be essential in the subsequent proof. Any (n, σ) partition π_2 can be constructed starting from any (n, σ) partition π_1 using increment/decrement operations. An increment/decrement operation on an (n, σ) partition is an operation in which one part of the partition is increased by 1 (the increment operation) while a second part is decreased by 1 (the decrement operation), resulting in another (n, σ) partition. In the case of the $(n, \sigma)_{\mathbf{M}}$ game, we represent an (n, σ) partition as a nondecreasingly ordered column of integers and we apply an increment or decrement operation to a specific row. Consider e.g. the $(5, 12)$ partitions $\pi_1 = (1, 1, 3, 3, 4)$ and $\pi_2 = (1, 1, 2, 3, 5)$ (for which $q_{12}^{\mathbf{M}} = 1/2$):

π_1	π_1		π_1	π_2
1	1		1	1
1	1		1	1
3	3	$\rightarrow$	3	2
3	3		3	3
4	4		4	5

We see that the increment operation is applied to row 5 and the decrement operation to row 3. For brevity, we say that row 5 is incremented and row 3 is decremented. In the present case, row 4 cannot be decremented instead of row 3, since the then obtained column of integers would no longer be nondecreasing. Through a concatenation of these increment/decrement operations, any (n, σ) partition π_2 can be obtained from the partition π_1. We can restrict these concatenations in the sense that once a row has been incremented (resp. decremented), it cannot be decremented (resp. incremented). Indeed, an increment operation followed later by a decrement operation (and vice versa) applied to the same row cancel each other out and can therefore be ignored. A concatenation of increment/decrement operations transforming π_1 into π_2 will be called a (π_1, π_2) transformation.

Let ν_i (resp. ν_d) denote the number of different incremented (resp. decremented) rows in the transformation of π_1 into π_2. Then we have that $Q_{\pi_1,\pi_2} > 1/2 \Leftrightarrow \nu_i < \nu_d$. This is easily seen by noting that ν_i (resp. ν_d) is nothing else but $\#\{j \mid i_j < i'_j\}$ (resp. $\#\{j \mid i_j > i'_j\}$). In general it thus holds that

$$Q_{\pi_1,\pi_2} = \frac{1}{2} - \frac{\nu_i - \nu_d}{2\,n}. \tag{10}$$

We illustrate (10) on some more examples.

Example 3.6.

(i) Consider the $(5, 18)$ partitions $\pi_1 = (2, 3, 4, 4, 5)$ and $\pi_2 = (1, 3, 3, 5, 6)$. The transformation of π_1 in π_2 goes (e.g.) as follows:

π_1	π_1		π_1	$\pi_1^{(1)}$		π_1	π_2
2	2		2	1		2	1
3	3		3	3		3	3
4	4	$\rightarrow$	4	4	$\rightarrow$	4	3
4	4		4	5		4	5
5	5		5	5		5	6

We obtain $\nu_i = \nu_d = 2$ and therefore $Q_{\pi_1,\pi_2} = \frac{1}{2}$.

(ii) Consider the $(5, 18)$ partitions $\pi_1 = (2, 3, 4, 4, 5)$ and $\pi_2 = (1, 1, 5, 5, 6)$. The transformation of π_1 in π_2 now goes (e.g.) as follows:

π_1	π_1		π_1	$\pi_1^{(1)}$		π_1	$\pi_1^{(2)}$		π_1	π_2
2	2		2	1		2	1		2	1
3	3	$\rightarrow$	3	3	$\rightarrow$	3	2	$\rightarrow$	3	1
4	4		4	4		4	4		4	5
4	4		4	5		4	5		4	5
5	5		5	5		5	6		5	6

Here, we obtain $\nu_i = 3$ and $\nu_d = 2$, which implies $Q_{\pi_1, \pi_2} = \frac{1}{2} - \frac{1}{10}$.

The above reasoning will be applied below. We discuss all (n, σ) partitions by considering three consecutive steps. The fourth step then determines the maximum value for the parts of an optimal strategy in an $(n, \sigma)_\mathbf{M}$ game.

Step 1: $n \leq 2$.

When $n = 1$ there is only one (n, σ) partition, when $n = 2$ it is obvious that all (n, σ) partitions play a draw. Indeed, for two $(2, \sigma)$ partitions $\pi_1 = (a_1, \sigma - a_1)$ and $\pi_2 = (b_1, \sigma - b_1)$, $a_1 \leq b_1$, it holds that either $\sigma - a_1 > \sigma - b_1$ when $a_1 < b_1$, or $\sigma - a_1 = \sigma - b_1$ when $a_1 = b_1$. The first two parts of Proposition 3.3 and (i) of Theorem 3.2 are therefore already proven. ∎

Step 2: Partitions satisfying

$$(\exists \, j > 1) \, (t_{j+1} > 0 \wedge n \geq 2 \, t_j + 3 + t_{j-1}). \tag{11}$$

These partitions are not optimal. Indeed, construct π_2 starting from π_1 by decrementing all t_j parts having value j by 1, decrementing a part having value $j + 1$ by two and incrementing $t_j + 2$ other parts from π_1, all different from $j - 1$. This transformation can be done using increment/decrement operations. The idea behind the transformation is that there will be two decrement operations applied to the row on which the first occurrence of $j + 1$ is situated in the original partition π_1, while all increment operations are applied to different rows. Using (10) we obtain that $Q_{\pi_1, \pi_2} = (n - 1)/(2n)$ and π_1 is therefore not optimal. Essential for this construction is that (11) holds, as this condition must be satisfied to be able to do all the increment operations on different rows. ∎

Example 3.7. Consider the $(8, 23)$ partition $\pi_1 = (1, 2, 2, 3, 3, 3, 4, 5)$. Condition (11) is satisfied for $j = 4$. If we choose $\pi_2 = (2, 3, 3, 3, 3, 3, 3, 3)$, we obtain $Q_{\pi_1, \pi_2} = \frac{n-1}{2n} = \frac{7}{16} < \frac{1}{2}$.

π_1	π_1		π_1	$\pi_1^{(1)}$		π_1	$\pi_1^{(2)}$		π_1	π_2
1	1		1	1		1	1		1	2
2	2		2	2		2	3		2	3
2	2		2	3		2	3		2	3
3	3	$\rightarrow$	3	3	$\rightarrow$	3	3	$\rightarrow$	3	3
3	3		3	3		3	3		3	3
3	3		3	3		3	3		3	3
4	4		4	3		4	3		4	3
5	5		5	5		5	4		5	3

In the last transformation, we see that the decremented part is again on the row where the first occurrence of $j + 1$ is situated in π_1, which is the reason why $Q_{\pi_1,\pi_2} < \frac{1}{2}$.

Step 3: All partitions not yet covered above are optimal.

These partitions satisfy

$$n \geq 3 \wedge (\forall j > 1) \ (t_{j+1} > 0 \Rightarrow n < 2\,t_j + 3 + t_{j-1}). \tag{12}$$

Before presenting the proof, we fix some notation. We say that an increment or decrement operation yields a decrementable (resp. incrementable) row, if after the increment or decrement operation a row becomes available for a decrement (resp. increment) operation and that row was not available before the increment or decrement operation was performed. Consider e.g. $\pi_1 = (2,3,3,5)$. It holds that incrementing row 3 yields an incrementable row (namely row 2) while incrementing row 4 does not yield an incrementable or decrementable row. Indeed, by incrementing row 3 we obtain $\pi_1' = (2,3,4,5)$ and in this partition row 2 is incrementable while it was not incrementable in partition π_1. Incrementing row 4 yields $\pi_1'' = (2,3,3,6)$ and all incrementable or decrementable rows are the same for π_1 and π_1''. We will also use the notions of first increment (resp. decrement) operation on a row and first increment (resp. decrement) operation on the same row. The former denotes an increment (resp. decrement) operation done on a row that has not yet been incremented or decremented in the process of transforming π_1 into π_2. The latter denotes the increment (resp. decrement) operation in the transformation step in which it happens for the first time that a row is incremented (resp. decremented) for a second time.

We now prove that all (n, σ) partitions π_1 satisfying (12) are optimal strategies. Suppose that there exists an (n, σ) partition π_2 that wins from π_1. Partition π_2 can again be obtained from partition π_1 using increment/decrement operations. From (10) we know that the number of incremented rows must be higher than the number of decremented rows. We now show that this implies that (11) holds, which contradicts (12).

Notice first that if (11) would be satisfied then there exists an (n, σ) partition π_2 that wins from π_1 such that there exists a (π_1, π_2) transformation in which the first decrement on the same row happens earlier than the first (if any) increment operation on the same row. Conversely, when there exists a (π_1, π_2) transformation such that the first decrement on the same row happens earlier than the first (if any) increment on the same row, then (11) must hold.

Since we suppose that π_1 is not optimal, the only case in which (11) would not be satisfied is when for all (n, σ) partitions π_2 that win from π_1, all possible (π_1, π_2) transformations would be such that the first increment on the same row happens earlier or at the same time as the first decrement on the same row. We therefore only need to show that a first increment on the same row is useless for obtaining rows that can be decremented and also for obtaining rows that can be incremented for the first time. In the next paragraph, we will show that an increment on the same row can only yield another row that has already been incremented. As the number of incremented rows must be higher than the number of decremented rows, it is therefore never necessary for the first increment on the

same row to happen earlier or at the same time as the first decrement on the same row.

It is obvious that, in general, not all rows can be used for an increment. For example, for the partition $\pi = (3, 3, 3)$ only the third row can be incremented. However, a first increment on a row can yield a row that can be incremented for the first time. For example, decrementing the first row and incrementing the last row of π results in $\pi' = (2, 3, 4)$. The increment of row 3 makes it possible to use row 2 of π' for an increment operation. This was impossible for partition π. A second increment on the same row, however, never yields a row that can be incremented for the first time. It is also obvious that an increment operation never yields a decrementable row.

As $Q_{\pi_1, \pi_2} < 1/2$, the above reasoning shows that there always exists a (π_1, π_2) transformation in which the second decrement on a certain row happens before the second increment (if any) on some other row. But this is impossible, since (12) would then not be satisfied. ∎

As Step 2 proved that all (n, σ) partitions, with $n \geq 3$, not satisfying (12) are not optimal, Step 3 proves that an (n, σ) partition, with $n \geq 3$, is optimal if and only if (12) is satisfied. ∎

Step 4: Determining a maximum value μ for the parts of an optimal strategy π_1 in any $(n, \sigma)_{\mathbf{M}}$ game with $n \geq 3$.

It can be easily verified that $\pi_1 = (2, 4, 6)$ is the only optimal strategy in the $(3, 12)_{\mathbf{M}}$ game, which implies $\mu > 5$.

First note that when an integer $j > 1$ exists such that $t_{j-1} = t_j = 0$ and $t_{j+1} \neq 0$, it holds that (12) is not satisfied for this value j and the partition therefore is not an optimal strategy. We can therefore assume $t_{j-1} = t_j = 0 \Rightarrow t_{j+1} = 0$, for any $j > 1$. This implies that $n \geq \lceil \frac{\mu}{2} \rceil$ and that there are at least $\lceil \frac{\mu}{2} \rceil \geq 3$ distinct parts in π_1. When $t_i \neq 0$ for all $2 \leq i \leq 6$, one can verify that (12) is not satisfied. Suppose therefore for some $1 < i < 6$ that $t_i = 0$ and $t_{i+1} \neq 0$, then it must hold that $n < 3 + t_{i-1}$. As there are at least 3 distinct parts, it holds that $n - t_{i-1} \geq 2$. This in turn implies that $n = 2 + t_{i-1}$, which implies that there are exactly three distinct numbers in the partition, implying $\mu \leq 6$. When $\mu = 6$, the fact that there are exactly three distinct numbers implies that $t_{2i-1} = 0$ and $t_{2i} > 0$, for $i \in \{1, 2, 3\}$. As $n < 3 + t_2$, $n < 3 + t_4$ and $n = t_2 + t_4 + t_6$ we obtain $t_2 = t_4 = t_6 = 1$, resulting in $\pi_1 = (2, 4, 6)$. Note that π_1 clearly satisfies (12).

Step 4 proves Lemma 3.1 and also the third part of Proposition 3.3. ∎

The above results can now be combined to prove Proposition 3.4. Indeed, from the above reasoning it follows that an (n, σ) partition π_1 is optimal if and only if either $n < 3$, or (12) holds. If $n \geq 3$ and $\pi_1 \neq (2, 4, 6)$, then we also know from the above reasoning that the optimal strategy contains no parts strictly greater than 5 and therefore has as multiplicity representation $(1^{t_1} 2^{t_2} 3^{t_3} 4^{t_4} 5^{t_5})$. We can now conclude the proof of Proposition 3.4 by making the following remarks. Firstly, it is obvious that

$$t_1 + t_2 + t_3 + t_4 + t_5 = n, \quad t_1 + 2t_2 + 3t_3 + 4t_4 + 5t_5 = \sigma, \tag{13}$$

is equivalent to saying that the partition is an (n, σ) partition and that it contains

no parts strictly greater than 5. Secondly, the three other conditions

$$
\begin{cases}
t_3 > 0 \Rightarrow t_2 + 2 > (t_3 - 1) + t_4 + t_5 \\
t_4 > 0 \Rightarrow t_3 + 2 > t_1 + (t_4 - 1) + t_5 \\
t_5 > 0 \Rightarrow t_4 + 2 > t_1 + t_2 + (t_5 - 1)
\end{cases}
$$

are merely a restatement of (12) using (13). This proves the third part of Proposition 3.4.

Theorem 3.2 now follows immediately.   ∎

4   Optimal strategies for $(n, \sigma)_\mathbf{L}$ games

4.1   Results

While not all $(n, \sigma)_\mathbf{P}$ and $(n, \sigma)_\mathbf{M}$ games have an optimal strategy, the situation is different for $(n, \sigma)_\mathbf{L}$ games.

Theorem 4.1. *All* $(n, \sigma)_\mathbf{L}$ *games have at least one optimal strategy.*

The exact characterization of these optimal strategies in an $(n, \sigma)_\mathbf{L}$ game is given by the following proposition.

Proposition 4.2. *Consider an* (n, σ) *partition* $\pi = (i_1, i_2, \ldots, i_n)$ *and let*

$$
a = \left\lfloor \frac{n}{2} \right\rfloor + 1 \, , \, b = \left\lfloor \frac{\sigma - n}{a} \right\rfloor + 1 \, , \, c =
\begin{cases}
n + 1 - \left\lfloor \frac{\sigma - n}{b - 1} \right\rfloor & , \text{ when } b \neq 1 \, , \\
n + 1 - (\sigma - n) & , \text{ when } b = 1 \, .
\end{cases}
\tag{14}
$$

The (n, σ) *partition* π *is an optimal strategy of an* $(n, \sigma)_\mathbf{L}$ *game if and only if one of the following four mutually exclusive conditions holds:*

(i) $\sigma - n \leq \lfloor n/2 \rfloor$ *and:*

-   $\pi = (1^{c-1} 2^{n-c+1})$.

(ii) $(n, \sigma) = (n, 2n)$, $n \geq 1$ *and:*

-   $\pi = (1^m 2^{(n-2m)} 3^m)$, $m \in \{0, 1, \ldots, \lfloor \frac{n}{2} \rfloor\}$.

(iii) $(n, \sigma) = (2l, \sigma)$, $l > 0$, $\sigma \neq 2n$, $\sigma > 3l$ *and:*

-   $(i_c = b \, \wedge \, \sigma \neq l(b + 2) + b - 1)$, *or*
-   $i_{l+1} \geq b + 1$, *or*
-   $\pi = (1^{l-1} b^2 (b + 1)^{l-1})$, *implying* $(n, \sigma) = (2l, l(b + 2) + b - 2)$.

(iv) $(n, \sigma) = (2l + 1, \sigma)$, $l \geq 0$, $\sigma \neq 2n$, $\sigma > 3l + 1$ *and:*

-   $i_c = b$, *or*
-   $\pi = (1^l b^1 (b + 1)^l)$, *implying* $(n, \sigma) = (2l + 1, l(b + 2) + b)$.

Example 4.3.

(i) The $(6, 17)_L$ game has 44 strategies of which 5 are optimal: $(1^3 4^2 6^1)$, $(1^3 4^1 5^2)$, $(1^2 2^1 4^2 5^1)$, $(1^2 3^1 4^3)$ and $(1^1 2^2 4^3)$. Note that for this game $b = 3$ and it therefore holds that $\sigma = l(b + 2) + b - 1$ (with $l = 3$), which implies that only the partitions for which $i_{l+1} \geq b + 1$ are optimal strategies.

(ii) The $(8, 23)_L$ game has 146 strategies and only one optimal strategy, given by $(1^3 4^5)$. Indeed, $b = c = 4$ and there are no strategies satisfying $i_{l+1} = b + 1$.

(iii) The $(9, 23)_L$ game has 123 strategies of which two are optimal: $(1^2 3^7)$ and $(1^4 3^1 4^4)$. As $b = c = 3$, the first partition corresponds to the case $i_c = b$ while the second one is of the form $(1^l b^1 (b + 1)^l)$.

We can also state the number of optimal strategies in function of $p(N, M, n)$. The function $p(N, M, n)$ is well known in partition theory and denotes the number of partitions of n into at most M parts, each smaller or equal to N [1]. By definition it holds that $p(N, M, 0) = 1$, and $p(N, M, n) = 0$ when $n < 0$. As there exists a generating function for $p(N, M, n)$, the numerical value of the number of optimal strategies can easily be obtained.

Proposition 4.4. *Let $p_n(M, N) = \sum_{i=0}^{N} p(N, M, N - i) p(N, n - M, i)$ and let*

$$\Sigma_1 = \sigma - n - \left\lfloor \frac{\sigma - n}{b - 1} \right\rfloor (b - 1), \quad \Sigma_2 = \sigma - l(b + 2), \tag{15}$$

with b and c defined in (14). The number of optimal strategies in an $(n, \sigma)_L$ game, here denoted as $v(n, \sigma)$, is then given in one of the following 5 mutually exclusive cases $(l > 0)$.

(i) $\sigma - n \leq \lfloor n/2 \rfloor \vee n = 1$:
$v(n, \sigma) = 1$.

(ii) $(n, \sigma) = (n, 2n) \wedge n > 1$:
$v(n, \sigma) = \lfloor \frac{n}{2} \rfloor + 1$.

(iii) $(n, \sigma) = (2l, \sigma) \wedge \sigma = l(b + 2) + b - 1 \wedge \sigma > 3l$:
$v(n, \sigma) = p_n(l, \Sigma_2)$.

(iv) $(n, \sigma) = (2l, \sigma) \wedge \sigma \neq 2n \wedge l(b + 2) + b - 1 > \sigma > 3l$:
$v(n, \sigma) = p_n(c - 1, \Sigma_1) + p_n(l, \Sigma_2) + \left\lceil \frac{l - c}{l + 1 - c} \right\rceil \left\lfloor \frac{\sigma}{l(b + 2) + b - 2} \right\rfloor$.

(v) $(n, \sigma) = (2l + 1, \sigma) \wedge \sigma \neq 2n \wedge \sigma > 3l + 1$:
$v(n, \sigma) = p_n(c - 1, \Sigma_1) + \left\lceil \frac{l + 1 - c}{l + 2 - c} \right\rceil \left\lfloor \frac{\sigma}{l(b + 2) + b} \right\rfloor$.

4.2 Proof

In the next five steps Theorem 4.1 and Proposition 4.2 are proven and thereafter, using the results from these five steps, Proposition 4.4 is proven.

Step 1: $\sigma - n \leq \lfloor n/2 \rfloor$.

We start by considering the special case of $(n, \sigma)_{\mathbf{L}}$ games for which $\sigma - n \leq \lfloor n/2 \rfloor$, corresponding to Case (i) of Proposition 4.2. Note that this condition is equivalent to $b = 1$. It is obvious that $\pi_1 = (1^{c-1}2^{n-c+1})$, with $c = n + 1 - (\sigma - n)$, is an (n, σ) partition and that π_1 wins from any other (n, σ) partition π_2. Indeed, let $k = \#\{j \mid i'_j = 1\}$ and $m = \#\{j \mid i'_j > 1\} = n - k$. As $\pi_2 \neq \pi_1$ it holds that $k \geq c > \lceil n/2 \rceil$ and $Q_{\pi_2, \pi_1} = (2m + (c - m - 1))/(2n) = (n - k + c - 1)/(2n) < 1/2$. In the remainder, we will assume $\sigma - n > \lfloor n/2 \rfloor$ and therefore $b > 1$. ∎

A partition will be represented graphically as a Ferrers graph, well known in the theory of partitions for visualizing a partition. Formally, it is the set of points with integral coordinates (j, k) in the plane such that if $\pi = (i_1, i_2, \ldots, i_n)$, then $(j, k) \in \mathcal{G}_\pi$ if and only if $0 \geq j \geq -n + 1$ and $0 \leq k \leq i_{|j|+1} - 1$ [1]. Although this representation is not essential in the proof, it helps to visualize the meaning of some variables that will be introduced.

For an (n, σ) partition π_1 we utilize the following values, which were already introduced in Proposition 4.2 (recall that $b > 1$):

$$a = \left\lfloor \frac{n}{2} \right\rfloor + 1, \quad b = \left\lfloor \frac{\sigma - n}{a} \right\rfloor + 1, \quad c = n + 1 - \left\lfloor \frac{\sigma - n}{b - 1} \right\rfloor. \qquad (16)$$

In words, b denotes the highest possible value for $i_{\lceil \frac{n}{2} \rceil}$ and c denotes the lowest possible value j such that $i_j = b$ is possible. Therefore, an (n, σ) partition π_1 for which $i_{\lceil \frac{n}{2} \rceil} = b$ surely exists.

Step 2: $n = 2l + 1 \wedge i_{l+1} < b$, or, $n = 2l \wedge i_l < b \wedge i_{l+1} < b + 1$.

When $n = 2l + 1$, any (n, σ) partition π_1 for which $i_{l+1} < b$ loses from any partition π_2 for which $i'_{l+1} = b$ and is therefore not an optimal strategy. Indeed, it then holds that $i'_{n-j} > i_{j+1}$, for any $0 \leq j \leq l$, which implies $Q_{\pi_2, \pi_1} \geq (l+1)/n > 1/2$. When $n = 2l$, then any partition π_1 for which $i_l < b$ and $i_{l+1} < b + 1$ loses from any partition π_2 for which $i'_l = b$. Indeed, it then holds that $i'_{n-j} > i_{j+1}$, for any $0 \leq j < l$ and $i'_l \geq i_{l+1}$, which again implies $Q_{\pi_2, \pi_1} > 1/2$. We can therefore already exclude these partitions π_1 as they are not optimal strategies. Note that this does not exclude a priori the possibility for an $(n, \sigma)_{\mathbf{L}}$ game to have optimal strategies. ∎

Step 3: $(n, \sigma) = (n, 2n)$.

All optimal strategies π_1 are given by

$$\pi_1 = (1^m 2^{n-2m} 3^m), m \in \{0, 1, \ldots, \lfloor n/2 \rfloor\}. \qquad (17)$$

We first prove that the strategies of type (17) are optimal. Let π_2 be another $(n, 2n)$ partition, with $k' = \#\{j \mid i'_j = 1\}$ and $m' = \#\{j \mid i'_j > 2\}$. As $\sigma = 2n$, we have that $k' \geq m'$. When $m' \leq m$, we obtain $Q_{\pi_2, \pi_1} \leq (2m + (n - 2m))/(2n) = 1/2$. When $m' > m$, we obtain $Q_{\pi_2, \pi_1} \leq (2m' + n - k' - m')/(2n) \leq 1/2$. We now

prove that the strategies (17) are the only optimal strategies in the $(n, 2n)_{\mathbf{L}}$ game. For any π_2, with k' and m' as defined above, such that $k' > m' > 0$, and π_1 with $m < m'$ it holds that $Q_{\pi_2,\pi_1} < 1/2$. When $m' = 0$ it holds that $\pi_2 = (2^n)$, which is of type (17). This proves Case (*ii*) of Proposition 4.2. ∎

We now subdivide the not yet covered $(n, \sigma)_{\mathbf{L}}$ games into those where n is even and those where n is odd.

Step 4: $n = 2l \wedge \sigma \neq 2n \wedge b \neq 1$.

Step 4.1: $i_{l+1} \geq b + 1$.

All (n, σ) partitions π_1 for which $i_{l+1} \geq b + 1$, if there are any, are optimal strategies. Indeed, suppose such a partition π_1 exists. For any (n, σ) partition π_2 it holds that $i'_l < b + 1$. Therefore, $i_{n-j} > i'_{j+1}$, for any $0 \leq j < l$, which implies $Q_{\pi_1,\pi_2} \geq 1/2$. This corresponds to the second part of Case (*iii*) of Proposition 4.2. ∎

Step 4.2: (n, σ) partitions satisfying

$$\sigma = l(b+2) + b - 1. \tag{18}$$

At least one (n, σ) partition π_1 satisfying $i_{l+1} \geq b + 1$ exists and these (n, σ) partitions comprise all optimal strategies. Indeed, any partition π_2 for which $i'_{l+1} < b + 1$ loses from the partition $\pi_1 = (1^{l-1}b^1(b+1)^l)$. This explains the condition $\sigma \neq l(b+2) + b - 1$ in the first part of Case (*iii*) of Proposition 4.2. ∎

Example 4.5. Consider the $(8, 27)_{\mathbf{L}}$ game, which has 352 strategies of which 10 are optimal, and for which (18) clearly holds ($b = \lfloor \frac{19}{5} \rfloor + 1 = 4$). The Ferrers graph for the partition $(1^3 b^1 (b+1)^4)$ is shown in Figure 2.
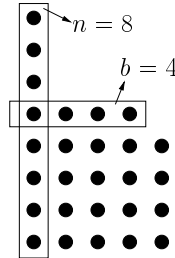
Figure 2: Ferrers graph for the $(8, 27)$ partition $(1, 1, 1, 4, 5, 5, 5, 5)$.

We now investigate the last remaining class of $(2l, \sigma)_{\mathbf{L}}$ games.

Step 4.3: $i_l = i_{l+1} = b \wedge \sigma \neq l(b+2) + b - 1 \wedge b \neq 1$.

The fact that (18) is not satisfied implies that for any two partitions π_1 satisfying $i_l = b$ and π_2 satisfying $i'_{l+1} \geq b + 1$ it holds that $Q_{\pi_1,\pi_2} = 1/2$. It therefore suffices to investigate Q_{π_1,π_2} with π_1 satisfying $i_l = i_{l+1} = b$ and π_2 satisfying $i'_l = i'_{l+1} = b$. Note that the strict inequality

$$\sigma < l(b+2) + b - 1 \tag{19}$$

must then hold. Indeed, $\sigma > l(b+2) + b - 1$ implies $b < \frac{\sigma - n + 1}{l+1} \leq \lfloor \frac{\sigma - n + l + 1}{l+1} \rfloor = b$, which is impossible.

We first introduce a useful lemma, considering both n even and n odd, which will make the subsequent proof and the proof of Step 5 simple.

Lemma 4.6. *For an (n, σ) partition π_1 with $i_{\lceil \frac{n}{2} \rceil} = i_{\lceil \frac{n}{2} \rceil + 1} = b > 1$ (b defined by (16)), let $s = \min\{j \mid i_j = b\}$ and $t = \max\{j \mid i_j = b\}$. It then holds that $t \geq n + 2 - s$ if $s < \lceil n/2 \rceil \wedge b > 2$, and $t \geq n + 1 - s$ if $s = \lceil n/2 \rceil \vee b = 2$.*

Let $n = 2l$ (resp. $n = 2l + 1$) when n is even (resp. odd). It holds that $c \leq s \leq \lceil n/2 \rceil$ (c defined by (16)) and $t \geq \lceil n/2 \rceil + 1$. By definition of s and t it must hold that

$$\sigma \geq s - 1 + (t - s + 1)b + (n - t)(b + 1), \tag{20}$$

or equivalently,

$$t \geq n - \sigma + nb - (s - 1)(b - 1). \tag{21}$$

First assume n is even. As (19) holds, we obtain (adding s to both sides of (21))

$$t + s > n - l(b + 2) - b + 1 + nb - (s - 1)(b - 1) + s,$$

which simplifies to

$$t + s > n + (l - s)(b - 2), \tag{22}$$

from which the desired inequalities immediately follow.

Now assume n is odd. From the tautology $b - 1 < b$, it follows that $\lfloor \frac{\sigma - n}{l+1} \rfloor < b$, which implies $\sigma - n < (l + 1)b$, finally implying $\sigma < l(b + 2) + b + 1$. Together with (21) this implies that

$$t + s > n + (l + 1 - s)(b - 2), \tag{23}$$

from which the desired inequalities again follow. ∎

Suppose $i'_c = i'_{l+1} = i_l = i_{l+1} = b$, with $n = 2l$. Let $r = \max\{j \mid i'_j = b\}$ and let s and t be defined as in the above lemma. Hence, the parts of π_1 and π_2 satisfy

$$\begin{cases} i_j < b, \text{ if } 1 \leq j < s, \\ i_j = b, \text{ if } s \leq j \leq t, \\ i_j > b, \text{ if } t < j \leq n, \end{cases} \qquad \begin{cases} i'_j < b, \text{ if } 1 \leq j < c, \\ i'_j = b, \text{ if } c \leq j \leq r, \\ i'_j > b, \text{ if } r < j \leq n. \end{cases}$$

It now holds that

$$\begin{aligned} Q_{\pi_2, \pi_1} &= \frac{1}{n} \left(\max(s - 1, n - r) + \right. \\ &\qquad \left. \frac{1}{2} \left(\min(n + 1 - c, t) - \max(s - 1, n - r) \right) \right) \\ &= \frac{1}{2n} \left(\min(n + 1 - c, t) + \max(s - 1, n - r) \right). \end{aligned}$$

First consider $s = c$. Using Lemma 4.6, we then obtain $Q_{\pi_2, \pi_1} = (n + 1 - c + s - 1)/(2n) = 1/2$. Partitions π_1 and π_2 satisfying $i_c = b$ resp. $i'_c = b$ therefore play a draw. Next, consider $s > c$, implying

$$Q_{\pi_2, \pi_1} = \frac{1}{2n} \left(\min(n + 1 - c, t) + s - 1 \right).$$

If $t \geq n + 1 - c$, then $Q_{\pi_2,\pi_1} = (n + s - c)/(2n)$, which implies $Q_{\pi_2,\pi_1} > 1/2$ for $s \neq c$. If $t < n + 1 - c$ then it holds that $Q_{\pi_2,\pi_1} = (t + s - 1)/(2n) \geq 1/2$ (again using Lemma 4.6). Moreover, when $s < l$ or $b > 2$, the same lemma implies $Q_{\pi_2,\pi_1} > 1/2$. The above already proves the first part of Case (*iii*) of Proposition 4.2.

Partitions satisfying $s = l$ and the case $b = 2$ need to be investigated further, to see if there are other optimal strategies possible. We therefore investigate when it holds that $Q_{\pi_2,\pi_1} = 1/2$, or equivalently when $t + s - 1 = n$. Inequality (20) is then equivalent to

$$\sigma \geq 2n + t(b - 2). \tag{24}$$

The definition of b from (16) implies $b(l + 1) > \sigma - n$ and combining this with (24), we obtain the strict inequality

$$(b - 2)(t - l) < b. \tag{25}$$

Inequality (25) is only satisfied when $b = 2$ or when $t = l + 1$ (recall that $b = 1$ is excluded and that $t \geq l + 1$). Indeed, when $t - l > 1$, it holds that (25) is equivalent to $b < 2 + 2/(t - l - 1)$, which can only hold when $b = 2$. When $b = 2$ it holds that $\sigma \geq 2n$ and the definition of b then implies $\sigma = 2n$ or $\sigma = 2n + 1$. The case $\sigma = 2n$ corresponds to Step 3 while $\sigma = 2n + 1$ implies that inequality (19) is not satisfied. When $t = l + 1$, we obtain $\sigma \geq 2n + (l + 1)(b - 2)$. When $\sigma > 2n + (l + 1)(b - 2)$, (19) is again not satisfied.

We now consider the case where $\sigma = 2n + (l + 1)(b - 2)$, $t = l + 1$ and $s = l$, implying that $\pi_1 = (1^{l-1}b^2(b + 1)^{l-1})$. We will prove that π_1 is optimal and therewith prove the third part of Case (*iii*) of Proposition 4.2. Consider another (n, σ) partition π_2 with $i'_l = i'_{l+1} = b$ and let $s' = \min\{j \mid i'_j = b\} \leq l$ and $t' = \max\{j \mid i'_j = b\} > l$. The parts of π_1 and π_2 then satisfy

$$\begin{cases} i_j < b, \text{ if } 1 \leq j < l, \\ i_j = b, \text{ if } l \leq j \leq l + 1, \\ i_j > b, \text{ if } l + 1 < j \leq n, \end{cases} \qquad \begin{cases} i'_j < b, \text{ if } 1 \leq j < s', \\ i'_j = b, \text{ if } s' \leq j \leq t', \\ i'_j > b, \text{ if } t' < j \leq n. \end{cases}$$

It follows that

$$Q_{\pi_2,\pi_1} = \frac{1}{n}\left((l - 1) + \frac{1}{2}\left(\min(t' - s' + 1, 2)\right)\right) = \frac{1}{2},$$

and therefore

$$\pi_1 = (1^{l-1}b^2(b + 1)^{l-1}) \tag{26}$$

is an optimal strategy. This corresponds to the third part of Case (*iii*) of Proposition 4.2.

The aggregation of the reasonings from Step 4 prove Case (*iii*) of Proposition 4.2. ∎

Example 4.7.

(i) Consider the $(6, 20)_{\mathbf{L}}$ game, for which it holds that $b = 4$ and $c = 3$. In Figure 3 the Ferrers graph of each optimal strategy of the game is given. The optimal strategies satisfying $i_c = b$ are given by $(1^1 3^1 4^4)$, $(2^2 4^4)$, $(1^1 2^1 4^3 5^1)$, $(1^2 4^2 5^2)$ and $(1^2 4^3 6^1)$. Note that the fourth partition is of type (26), but as $c = l$ it is not a special case. As can be easily seen in the Ferrers graphs, these optimal strategies differ from each other by rearranging the $\Sigma_1 = 2$ dots that can be moved around freely. The remaining optimal strategies are those satisfying $i_{l+1} \geq b + 1$, given by $(1^2 3^1 5^3)$, $(1^1 2^2 5^3)$, $(1^2 2^1 5^2 6^1)$, $(1^3 5^1 6^2)$ and $(1^3 5^2 7^1)$. These latter optimal strategies differ from each other by rearranging the $\Sigma_2 = 2$ free dots. Note that Σ_1 and Σ_2 are defined by (15).
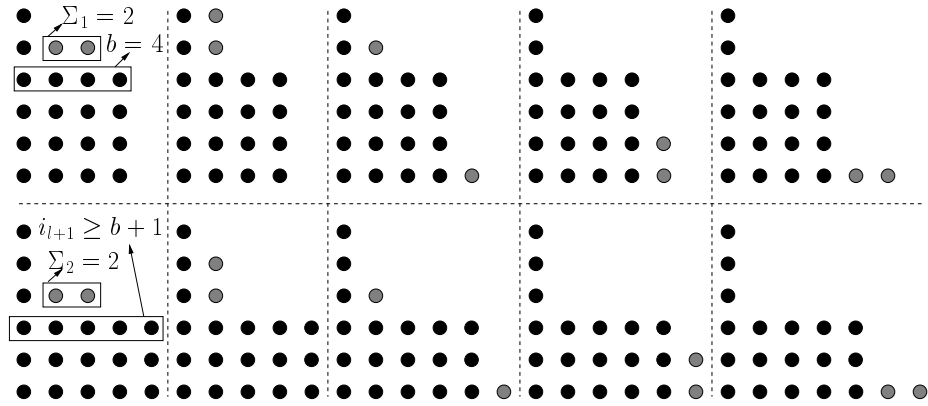
Figure 3: Optimal strategies of the $(6, 20)_{\mathbf{L}}$ game.

(ii) Consider the $(12, 32)_{\mathbf{L}}$ game. We obtain that $b = 3$, $c = 3$, $l(b+2) + b - 1 = 32 = \sigma$. The only $(12, 32)$ partition satisfying $i_c = b$ is $\pi_1 = (1^2 3^{10})$ and when $\pi_2 = (1^5 3^1 4^6)$ it indeed holds that $Q_{\pi_2, \pi_1} > 1/2$. All optimal strategies are therefore those satisfying $i_{l+1} \geq b + 1$, given by $(1^5 3^1 4^6)$, $(1^4 2^2 4^6)$, $(1^5 2^1 4^5 5^1)$, $(1^6 4^5 6^1)$ and $(1^6 4^4 5^2)$.

(iii) Consider the $(14, 36)_{\mathbf{L}}$ game. We now obtain that $b = 3$, $c = 4$, $l(b+2) + b - 1 = 37 > \sigma$. There is one $(14, 36)$ partition satisfying $i_c = b$, namely $(1^3 3^{11})$, and it is an optimal strategy. The other optimal strategies all satisfy $i_{l+1} \geq b + 1$, and are given by $(1^6 2^1 4^7)$ and $(1^7 4^6 5^1)$.

(iv) Consider the $(4, 22)_{\mathbf{L}}$ game. For the above 3 examples the optimal strategies satisfying $i_{l+1} \geq b$ always satisfied $i_{l+1} = b + 1$. In general this is not true, as is indicated by the present example, for which it holds that $b = 7$ and for which the optimal strategy $(1^2 (10)^2)$ satisfies $i_{l+1} > b + 1$. We do not explicitly specify the other optimal strategies for this game, as they are numerous.

Step 5: $n = 2l + 1 \wedge \sigma \neq 2n \wedge b \neq 1$.

All (n, σ) partitions π_1 for which $i_c = b$ or for which $i_{l+1} = b \wedge i_{l+2} = b + 1$ are the only optimal strategies. The proof is completely analogous to the proof for n even. It follows directly that all optimal strategies π_1 must satisfy the condition $i_{l+1} = b$, and that such a strategy always exists. Secondly, it is evident that partitions of type

$$\pi_1 = (1^l b^1 (b+1)^l) \tag{27}$$

are optimal strategies and these only exist in $(2l + 1, l(b+2) + b)_{\mathbf{L}}$ games. Finally, using Lemma 4.6, we obtain in a completely analogous way as in Step 4.3 that partitions of type (27) are the only possible optimal strategies that do not satisfy $i_c = b$, and that all (n, σ) partitions that satisfy $i_c = b$ are optimal. This proves Case (iv) of Proposition 4.2. Note that if $c = l + 1$, partition (27) satisfies $i_c = b$ and is then not a special case. ∎

Example 4.8. Consider the $(7, 18)_{\mathbf{L}}$ game. We obtain that $b = c = 3$ and $\sigma = l(b+2) + b$. The optimal strategies are therefore given by $(1^3 3^1 4^3)$, $(1^2 3^4 4^1)$ and $(1^1 2^1 3^5)$, the first one being of type (27).

As the cases above covered all possible $(n, \sigma)_{\mathbf{L}}$ games and for each game there was always at least one optimal strategy, we have also proven Theorem 4.1. ∎

Using the above descriptions of the optimal strategies, we can state the number of optimal strategies for any $(n, \sigma)_{\mathbf{L}}$ game using the function $p_n(M, N) = \sum_{i=0}^{N} p(N, M, N - i) p(N, n - M, i)$, which was already introduced. Proposition 4.4 is proven below, using the previously introduced values b and c, defined by (16), and Σ_1 and Σ_2 defined by (15). The number of optimal strategies in an $(n, \sigma)_{\mathbf{L}}$ game, here denoted as $v(n, \sigma)$, is then given by:

(i) $\sigma - n \leq \lfloor n/2 \rfloor \vee (n, \sigma) = (1, \sigma)$:
$v(n, \sigma) = 1$.

When $n = 1$ there is only one strategy, namely (σ). The result for $\sigma - n \leq \lfloor n/2 \rfloor$, which is equivalent to $b = 1$, follows from the result of Step 1.

(ii) $(n, \sigma) = (2, \sigma)$:
$v(n, \sigma) = \lfloor \frac{\sigma}{2} \rfloor$.

All strategies are optimal, this follows implicitly from the proofs of this subsection and this case is implicitly included in Proposition 4.4.

(iii) $(n, \sigma) = (n, 2n)$:
$v(n, \sigma) = \lfloor \frac{n}{2} \rfloor + 1$.

This is immediately clear by counting the optimal strategies obtained in Step 3.

(iv) $(n, \sigma) = (2l, \sigma) \wedge \sigma = l(b+2) + b - 1 \wedge \sigma \neq 2n \wedge b \neq 1$:
$v(n, \sigma) = p_n(l, \Sigma_2)$.

This corresponds to Step 4.2. We have to count the number of (n, σ) partitions for which $i_{l+1} \geq b + 1$. We can construct all of them by starting with the Ferrers graph of $(1^l (b+1)^l)$ and distributing the remaining Σ_2 dots in all

possible combinations to obtain all Ferrers graphs of (n, σ) partitions with $i_{l+1} \geq b + 1$ (cfr. the bottom row of Figure 3).

(v) $(n, \sigma) = (2l, \sigma) \wedge \sigma = l(b + 2) + b - 2 \wedge \sigma \neq 2n \wedge b \neq 1$:
$v(n, \sigma) = p_n(c - 1, \Sigma_1) + p_n(l, \Sigma_2) + \lceil \frac{l-c}{l+1-c} \rceil$.

This corresponds to Steps 4.1 and 4.3, in the case that (26) is a possible strategy. Here, we have to count the number of (n, σ) partitions for which $i_c = b$, this is given by $p_n(c - 1, \Sigma_1)$. We also have to count the number of (n, σ) partitions for which $i_{l+1} \geq b + 1$, given by $p_n(l, \Sigma_2)$. Finally we also have to take into account the special case (26). Unless $c = l$, this partition has not yet been counted.

(vi) $(n, \sigma) = (2l, \sigma) \wedge \sigma < l(b + 2) + b - 2 \wedge \sigma \neq 2n \wedge b \neq 1$:
$v(n, \sigma) = p_n(c - 1, \Sigma_1) + p_n(l, \Sigma_2)$.

This corresponds to Steps 4.1 and 4.3, when (26) is not a possible strategy. This case and the previous case are combined into Case (*iv*) of Proposition 4.4.

(vii) $(n, \sigma) = (2l + 1, \sigma) \wedge \sigma = l(b + 2) + b \wedge \sigma \neq 2n \wedge b \neq 1$:
$v(n, \sigma) = p_n(c - 1, \Sigma_1) + \lceil \frac{l+1-c}{l+2-c} \rceil$.

This corresponds to Step 5, in the case that (27) is a possible strategy. Here, we have to count the number of (n, σ) partitions for which $i_c = b$ and also the special partition $(1^l b^1 (b + 1)^l)$, which has not yet been counted unless $c = l + 1$.

(viii) $(n, \sigma) = (2l + 1, \sigma) \wedge \sigma < l(b + 2) + b \wedge \sigma \neq 2n \wedge b \neq 1$:
$v(n, \sigma) = p_n(c - 1, \Sigma_1)$.

This corresponds to Step 5, when (27) is not a possible strategy. The current case and the previous case are combined into Case (*v*) of Proposition 4.4. ∎

5   Conclusion

We have introduced three interesting variants of the same game, played with partitions of σ into n parts, σ and n fixed before the game starts. The games are defined by viewing a partition as a random variable uniformly distributed over the parts of the partition and by stochastically comparing these random variables. The definitions of the game variants differ from each other only by the copula used to couple the marginal uniform cdf into the bivariate cdf. For each game variant, we have characterized which games possess optimal strategies and we explicitly stated these strategies. It is clear from the results that the optimal strategies are characterized completely differently for each game variant.

We conclude by giving four tables containing the number of (n, σ) partitions and the number of optimal strategies in the $(n, \sigma)_{\mathbf{M}}$, resp. $(n, \sigma)_{\mathbf{P}}$, resp. $(n, \sigma)_{\mathbf{L}}$ game (for $n \in [1, 25]$ and $\sigma \in [n, n + 24]$). The row number denotes the value n, while the column number denotes the value $\sigma - n$. For example, the number of optimal strategies of the $(8, 27)_{\mathbf{L}}$ game, is located in Table 4 in row number 8

and column number $27 - 8 = 19$ and is given by 10. As can be deduced from the tables, the most interesting games are those where $\sigma > 2n > 4$.

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Table 1: Number of (n, σ) partitions.

$\sigma \backslash n$ / $n \backslash \sigma$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9	9	10	10	11	11	12	12	13
3	1	1	2	3	4	5	7	8	10	12	14	16	19	21	24	27	30	33	37	40	44	48	52	56	61
4	1	1	2	3	5	6	9	11	15	18	23	27	34	39	47	54	64	72	84	94	108	120	136	150	169
5	1	1	2	3	5	7	10	13	18	23	30	37	47	57	70	84	101	119	141	164	192	221	255	291	333
6	1	1	2	3	5	7	11	14	20	26	35	44	58	71	90	110	136	163	199	235	282	331	391	454	532
7	1	1	2	3	5	7	11	15	21	28	38	49	65	82	105	131	164	201	248	300	364	436	522	618	733
8	1	1	2	3	5	7	11	15	22	29	40	52	70	89	116	146	186	230	288	352	434	525	638	764	919
9	1	1	2	3	5	7	11	15	22	30	41	54	73	94	123	157	201	252	318	393	488	598	732	887	1076
10	1	1	2	3	5	7	11	15	22	30	42	55	75	97	128	164	212	267	340	423	530	653	807	984	1204
11	1	1	2	3	5	7	11	15	22	30	42	56	76	99	131	169	219	278	355	445	560	695	863	1060	1303
12	1	1	2	3	5	7	11	15	22	30	42	56	77	100	133	172	224	285	366	460	582	725	905	1116	1380
13	1	1	2	3	5	7	11	15	22	30	42	56	77	101	134	174	227	290	373	471	597	747	935	1158	1436
14	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	175	229	293	378	478	608	762	957	1188	1478
15	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176	230	295	381	483	615	773	972	1210	1508
16	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176	231	296	383	486	620	780	983	1225	1530
17	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176	231	297	384	488	623	785	990	1236	1545
18	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176	231	297	385	489	625	788	995	1243	1556
19	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176	231	297	385	490	626	790	998	1248	1563
20	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176	231	297	385	490	627	791	1000	1251	1568
21	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176	231	297	385	490	627	792	1001	1253	1571
22	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176	231	297	385	490	627	792	1002	1254	1573
23	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176	231	297	385	490	627	792	1002	1255	1574
24	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176	231	297	385	490	627	792	1002	1255	1575
25	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176	231	297	385	490	627	792	1002	1255	1575

σ / n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9	9	10	10	11	11	12	12	13
3	1	1	2	2	3	3	2	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	1	1	2	2	3	4	3	2	2	3	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	1	1	2	2	3	3	4	4	3	3	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0
6	1	1	2	2	3	3	5	5	3	4	2	1	2	1	0	0	0	0	0	0	0	0	0	0	0
7	1	1	2	2	3	3	3	4	6	5	4	5	3	1	1	1	0	0	0	0	0	0	0	0	0
8	1	1	2	2	3	3	3	4	4	6	6	5	6	4	2	2	1	1	0	0	0	0	0	0	0
9	1	1	2	2	3	3	3	4	4	4	7	7	5	7	5	3	3	1	0	0	0	0	0	0	0
10	1	1	2	2	3	3	3	4	4	4	5	8	7	6	8	6	4	4	2	1	1	1	0	0	0
11	1	1	2	2	3	3	3	4	4	4	5	5	8	8	7	9	7	5	5	3	1	1	0	0	0
12	1	1	2	2	3	3	3	4	4	4	5	5	9	9	7	10	8	6	7	4	2	2	1	0	0
13	1	1	2	2	3	3	3	4	4	4	5	5	5	6	10	9	8	11	9	7	8	5	3	3	1
14	1	1	2	2	3	3	3	4	4	4	5	5	5	6	6	10	10	9	12	10	8	9	7	4	4
15	1	1	2	2	3	3	3	4	4	4	5	5	5	6	6	6	11	11	9	13	11	9	11	8	5
16	1	1	2	2	3	3	3	4	4	4	5	5	5	6	6	6	7	12	11	10	14	12	10	12	9
17	1	1	2	2	3	3	3	4	4	4	5	5	5	6	6	6	7	7	12	12	11	15	13	11	13
18	1	1	2	2	3	3	3	4	4	4	5	5	5	6	6	6	7	7	7	13	13	11	16	14	12
19	1	1	2	2	3	3	3	4	4	4	5	5	5	6	6	6	7	7	7	8	14	13	12	17	15
20	1	1	2	2	3	3	3	4	4	4	5	5	5	6	6	6	7	7	7	8	8	14	14	13	18
21	1	1	2	2	3	3	3	4	4	4	5	5	5	6	6	6	7	7	7	8	8	8	15	15	13
22	1	1	2	2	3	3	3	4	4	4	5	5	5	6	6	6	7	7	7	8	8	8	9	16	15
23	1	1	2	2	3	3	3	4	4	4	5	5	5	6	6	6	7	7	7	8	8	8	9	9	16
24	1	1	2	2	3	3	3	4	4	4	5	5	5	6	6	6	7	7	7	8	8	8	9	9	9
25	1	1	2	2	3	3	3	4	4	4	5	5	5	6	6	6	7	7	7	8	8	8	9	9	9

Table 2: Number of optimal strategies for $(n, \sigma)_{\mathbf{M}}$ games.

σ / n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9	9	10	10	11	11	12	12	13
3	1	1	1	2	1	1	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	1	1	1	1	3	1	1	0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0
5	1	1	1	1	1	3	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0
6	1	1	1	1	1	1	4	1	1	0	1	0	2	0	0	0	1	0	0	0	1	0	0	0	0
7	1	1	1	1	1	1	1	4	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0
8	1	1	1	1	1	1	1	1	5	1	1	0	1	0	1	0	2	0	0	0	0	0	0	0	1
9	1	1	1	1	1	1	1	1	1	5	0	0	0	0	0	0	0	0	2	0	1	0	0	0	1
10	1	1	1	1	1	1	1	1	1	1	6	1	1	0	1	0	1	0	1	0	2	0	0	0	0
11	1	1	1	1	1	1	1	1	1	1	1	6	0	0	0	0	0	0	0	0	0	0	2	0	0
12	1	1	1	1	1	1	1	1	1	1	1	1	7	1	1	0	1	0	1	0	1	0	1	0	3
13	1	1	1	1	1	1	1	1	1	1	1	1	1	7	0	0	0	0	0	0	0	0	0	0	0
14	1	1	1	1	1	1	1	1	1	1	1	1	1	1	8	1	1	0	1	0	1	0	1	0	1
15	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	8	0	0	0	0	0	0	0	0	0
16	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	9	1	1	0	1	0	1	0	1
17	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	9	0	0	0	0	0	0	0
18	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	10	1	1	0	1	0	1
19	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	10	0	0	0	0	0
20	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	11	1	1	0	1
21	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	11	0	0	0
22	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	12	1	1
23	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	12	0
24	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	13
25	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Table 3: Number of optimal strategies for $(n, \sigma)_{\mathbf{P}}$ games.

σ / n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9	9	10	10	11	11	12	12	13
3	1	1	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1
4	1	1	1	1	3	2	2	4	5	3	7	8	6	10	14	9	16	20	15	22	30	21	32	40	31
5	1	1	1	1	1	3	1	2	2	1	2	5	1	2	5	1	2	5	1	2	5	1	2	5	1
6	1	1	1	1	1	1	4	2	1	3	4	5	2	4	10	10	3	7	15	18	6	12	23	30	11
7	1	1	1	1	1	1	1	4	1	2	1	3	1	2	5	2	1	2	5	10	1	2	5	10	1
8	1	1	1	1	1	1	1	1	5	2	1	2	2	5	5	1	3	7	7	10	2	4	10	20	20
9	1	1	1	1	1	1	1	1	1	5	1	2	1	2	2	1	2	5	1	3	1	2	5	10	2
10	1	1	1	1	1	1	1	1	1	1	6	2	1	2	1	3	4	5	1	2	6	3	8	10	1
11	1	1	1	1	1	1	1	1	1	1	1	6	1	2	1	2	1	3	1	2	5	1	2	6	1
12	1	1	1	1	1	1	1	1	1	1	1	1	7	2	1	2	1	2	2	5	5	1	2	5	2
13	1	1	1	1	1	1	1	1	1	1	1	1	1	7	1	2	1	2	1	2	2	1	2	5	1
14	1	1	1	1	1	1	1	1	1	1	1	1	1	1	8	2	1	2	1	2	1	3	4	5	1
15	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	8	2	1	2	1	2	1	2	3	1
16	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	9	2	1	2	1	2	1	2	2
17	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	9	1	2	1	2	1	2	1
18	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	10	2	1	2	1	2	1
19	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	10	1	2	1	2	1
20	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	11	2	1	2	1
21	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	11	1	2	1
22	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	12	2	1
23	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	12	1
24	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	13
25	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Table 4: Number of optimal strategies for $(n, \sigma)_{\mathbf{L}}$ games.