Constructions of subgeometry partitions

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Abstract

New constructions of Sperner spaces using 'generalized extended André spreads' are used to construct a wide variety of new subgeometry partitions of finite projective spaces.

1 Introduction

Let Ω be a projective space and let Π be a subset of points of Σ such that if a line ℓ of Ω intersects Π in at least two points A and B then we define a 'line' of Π to be the set of points $\ell \cap \Pi$. If the points of Π and the 'lines' induced from lines of Σ form a projective space, we say that Π is a 'subgeometry' of Σ. A 'subgeometry partition' of a projective space Ω is a partition of Ω into mutually disjoint subgeometries.

Subgeometry partitions originally arose in the context of André planes as follows: Suppose we have a projective space $PG(3,q^2)$ that is covered by subgeometries isomorphic to either $PG(3,q)$ or $PG(1,q^2)$. Bruen and Thas [3] found some examples of such subgeometry partitions and further provided a construction technique that we call here 'geometric lifting' that produces a translation plane of order q^4 with spread in $PG(7,q)$. The particular examples turned out to be André planes of order q^4. The question then became can this construction process be reversed? Another way to think about this is to ask what sort of property of the André planes that guarantees that such a plane has been geometrically lifted from a subgeometry partition? More generally, Hirschfeld and Thas [5] consider such a geometric lifting process from Baer subgeometry partitions ($PG(n,q^2)$ partitioned by $PG(n,q)$'s) that produces translation planes of order q^{2n+1} and the same question as to the nature of the translation plane may be asked.

In Johnson [8], both these question are answered as follows: Given any translation plane of finite order with $GF(q)$ in the kernel, if there exists a fixed point free

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collineation group G in the translation complement of order q^2-1 that contains the kernel homology group of order $q-1$, then the translation plane 'retracts' to a subgeometry partition. If the order of the plane is q^4 then a partition of $PG(3,q^2)$ by subgeometries isomorphic to $PG(3,q)$ and $PG(1,q^2)$ may be constructed. But the retraction process is valid for translation planes of any order q^n: If n is even a subgeometry partition of $PG(n-1,q^2)$ by subgeometries isomorphic to $PG(n-1,q)$ and $PG(n/2-1,q^2)$. If n is odd a subgeometry partition of $PG(n-1,q^2)$ by $PG(n-1,q)$'s is obtained. In the latter case, the subgeometry partition is said to be a 'Baer subgeometry partition', and in the former case, the partition is called a 'mixed partition'.

Now, as mentioned, the André planes of order q^n admit a collineation group of order q^n-1 that contains the sub-kernel homology group of order $q-1$ and is necessarily fixed-point-free. Hence, if n is even, retraction of any André plane of square order produces a mixed subgeometry partition.

Since the action of the group of order q^2-1 is the active ingredient in the retraction process, it is a natural question to ask if one would use other groups of orders q^k-1 with similar properties, whether subgeometry partitions of projective spaces with different subgeometries are possible to construct from translation planes. In Johnson [10], this idea was furthered obtaining what are called 'quasi-subgeometry' partitions.

Definition 1. *A 'quasi-subgeometry' of a projective space is a subset of points that can be made into a projective space such that the lines of the projective set are subsets of lines of the projective space.*

It was shown in Johnson [10], that any translation plane of order q^n with spread in $PG(2n-1,q)$ that admits a fixed-point-free group of order q^k-1, for k a divisor of n, containing the kernel homology group does actually produce quasi-subgeometry partitions of an associated projective space with a wide variety of possible quasi-subgeometry types involved in the partition. The following result from Johnson [10], details the possibilities:

Corollary 1. *(Johnson [10]) Assume a partial spread $\mathcal{Z}$ of order q^{ds} in a vector space of dimension $2ds$ over K isomorphic to $GF(q)$ admits a fixed-point-free field group F_w^* of order (q^w-1) containing K^*, for $w=d$ or $2d$ and s is odd if $w=2d$. Then any component orbit Γ of length $(q^w-1)/(q^e-1)$ (a 'q^e-fan'), for e a divisor of d, produces a quasi-subgeometry isomorphic to a $PG(ds/e-1,q^e)$ in the corresponding projective space $PG(2ds/w-1,q^w)$, considered as the lattice of F_w-subspaces of V.*

These results are also valid in 'generalized spreads' in the infinite case, over skewfields, and in the so called t-spreads.

Definition 2. *A 't-spread' of a finite vector space V over $GF(q)$ is a partition of the non-zero vectors by t-dimensional $GF(q)$-subspaces. Any t-spread of V induces a corresponding partition of the associated projective space of lattice of vector subspaces by projective $t-1$spaces.*

We use the term 't-spread' in both the vector space and projective space settings.

In fact, general 'subgeometry partitions of projective spaces' (that is, quasi-subgeometry partitions that actually turn out to be subgeometry partitions) generally do not produce translation plane type spreads, but always produce t-spreads of vector spaces, for an appropriate value of t. The problem of ascertaining whether a quasi-subgeometry partition is actually a subgeometry partition has been essentially unresolved. For example, since André planes of order q^6 admit fixed-point-free groups of order $q^3 - 1$ containing the kernel subgroup of order $q - 1$, such planes always produce quasi-subgeometry partitions, but it is an open question whether these partitions are actually subgeometry partitions.

In a previous article Johnson [11], the concept of an André plane as arising from a Desarguesian affine plane is generalized to analogous structures constructible from what are called 'Desarguesian t-spreads'. These structures produce Sperner spaces from Desarguesian t-spreads in a similar way that André planes are produced from Desarguesian affine planes. In essentially all of the Sperner spaces that are constructed, it is possible to produce a wide variety of subgeometry geometry partitions using groups of order $q^k - 1$. Also, we are able to obtain subgeometry partitions with a large number of mutually non-isomorphic subgeometries (that is, many more than simply one or two types).

Some of the subgeometry partitions that we construct here were first constructed by Ebert and Mellinger in [4]. However, in that paper, Ebert and Mellinger first construct the subgeometry partition and then 'geometrically lift' to associated t-spreads. Furthermore, the subgeometry partitions constructed are mixed and have exactly two types of subgeometries in the partition.

In our treatment, we normally begin in the affine setting (vector space setting) and consider first the t-spread, look for an appropriate collineation group that allows a 'retraction', thereby constructing the subgeometry partitions using such fixed-point-free groups. Moreover, this approach allows a complete generalization of the 'lifting' technique, which we call 'unfolding the fan' and 'retraction', which we call 'folding the fan' and leads to a great variety of mutually non-isomorphic subgeometries with a great mixture of possible types of subgeometries.

2 Background

Previously, partitions of projective spaces by 'quasi-subgeometries' has been considered. This background section is given so that the reader might better understand when a quasi-subgeometry becomes a 'subgeometry' and the partitions then becomes subgeometry partitions.

Most of the construction techniques have been previously given in Johnson [10]. As this material is quite general and suitable both for finite and infinite affine translation planes and more generally for generalized spreads, we repeat the main part of these results here.

Definition 3. *A 'q^e-fan' in an rds-dimensional vector space over K isomorphic to $GF(q)$ is a set of $(q^w - 1)/(q^e - 1)$ mutually disjoint K-subspaces of dimension ds that are in an orbit under a field group F_w^* of order $(q^d - 1)$, where F_w contains K and such that F_w^* is fixed-point-free, and where $w = d$ or $2d$ and in the latter case s is odd.*

We use the term that $GF(q^w)$ 'acts' on the vector space, if the multiplicative group acts in a manner as in the previous definition.

Theorem 1. *(Johnson [10]) (1) Let $\mathcal{S}$ be a t-spread of V of size (ds, q), and assume that $w = d$ or t^*d where t^*d does not divide ds, and t^* divides t.*

Assume that $GF(q^w)$ 'acts' on $\mathcal{S}$ then there is a quasi-subgeometry partition of $PG(tds/w - 1, q^w)$ consisting of $PG(ds/e_i - 1, q^{e_i})$'s where e_i divides w for $i \in \Lambda$.

(2) Conversely, any quasi-subgeometry partition of $PG(tds/w - 1, q^w)$ by $PG(ds/e_i - 1, q^{e_i})$'s for $i \in \Lambda$, and e_i a divisor of w, produces a t-spread of size (ds, q) that is a union of q^{e_i}-fans. The t-spread corresponds to a translation Sperner space admitting $GF(q^w)^*$ as a collineation group.

Remark 1. *All of the constructions above involve 'fans'. A 'fan' produces a quasi-subgeometry and involves a process that is called 'folding the fan'. A quasi-subgeometry also produces a 'fan', wherein the process of is called 'unfolding the fan'. In other terminology, 'folding the fan' has been called 'retraction' in Johnson [8], and 'unfolding the fan' is called 'lifting' in Hirschfeld and Thas [5] and Ebert and Mellinger [4].*

2.1 $r - (sn, q)$-Spreads

The following background information is taken from Johnson [11].

Consider a field $GF(q^{rsn})$, where $q = p^z$, for p a prime. Then $GF(q^{rsn})$ is an r-dimensional vector space over $GF(q^{sn})$. More generally, let V be the r-dimensional vector space over $GF(q^{sn})$.

Definition 4. *A '1-dimensional r-spread' is defined to be a partition of V by the set of all 1-dimensional $GF(q^{sn})$-subspaces, where q is a prime power.*

In this case, the vectors are represented in the form $(x_1, x_2, ..., x_r)$, where $x_i \in GF(q^{sn})$.

Definition 5. *Furthermore, the 1-dimensional $GF(q^{sn})$-subspaces may be partitioned in the following sets called 'j-(0-sets)'. A 'j-(0-set)'is the set of vectors with j of the entries equal to 0. For a specific set of j-zeros among the r elements, the set of such non-zero vectors in the remaining $r - j$ non-zero entries is called a '(j-(0-subset))'.*

Note that there are exactly $\binom{r}{r-j}(q^{sn} - 1)^{r-j}$ vectors (non-zero vectors) in each j-(0-set) and exactly $(q^{sn} - 1)^{r-j}$ vectors in each of the $\binom{r}{r-j}$ disjoint $j-$(0-subsets).

Notation 1. *Hence, $j = 0, 1, ..., r - 1$ and we denote the $j -$ (0-sets) by Σ_j and by not specifying any particular order, we index the $\binom{r}{r-j}$ $j -$ (0-subsets) by $\Sigma_{j,w}$, for $w = 1, 2, ..., \binom{r}{r-j}$. We note that*

$$\cup_{w=1}^{\binom{r}{r-j}} \Sigma_{j,w} = \Sigma_j,$$

a disjoint union.

Remark 2. *Furthermore, the* $(q^{rsn}-1)$ *non-zero vectors are partitioned in the* $j-(0\text{-sets})$ *by*

$$(q^{rsn}-1)=\sum_{j=0}^{r-1}\binom{r}{r-j}(q^{sn}-1)^{r-j},$$

and the number of 1*-dimensional* $GF(q^{sn})$*-subspaces is*

$$(q^{rsn}-1)/(q^{sn}-1)=\sum_{j=0}^{r-1}\binom{r}{r-j}(q^{sn}-1)^{r-j-1}.$$

Also, note that this is then also the number of sn*-dimensional* $GF(q)$*-subspaces in a* $r-(sn,q)$*-spread.*

Notation 2. *Consider a vector* $(x_1,x_2,...,x_r)$ *over* $GF(q^{sn})$*, we use the notation* (x_1,y) *for this vector. Consider a* j*-(*0*-set)* Σ_j *and let* x_{j_1} *denote the first non-zero entry. Then all of the other entries are of the form* $x_{j_1}m$*, for* $m\in GF(q^{sn})$*. For example, the elements of an element of a* 0*-(*0*-set) may be presented in the form* $(x_1,x_1m_1,...,x_1m_{r-1})$*, for* x_1 *non-zero and* m_i *also non-zero in* $GF(q^{sn})$*. That is,* $y=(x_1m_1,...,x_1m_{r-1})$*. More importantly, if we vary* x_1 *over* $GF(q^{sn})$*, then*

$$y=(x_1m_1,...,x_1m_{r-1}),$$

is a 1*-dimensional* $GF(q^{sn})$*-subspace. However, we now consider this subspace as an* sn*-dimensional* $GF(q)$*-subspace.*

In this notation, a Desarguesian 1-spread leads to an affine translation plane by defining 'lines' to be translates of the 1-dimensional $GF(q^{sn})$-subspaces.

Definition 6. *In general, for* $r>2$*, a Desarguesian* r*-spread leads to a 'Desarguesian translation Sperner space' by the same definition on lines. Every* 1*-dimensional* $GF(q^{sn})$*-space may be considered a* sn*-space over* $GF(q)$*. When this occurs we have what we shall call an '*r*-(*sn*-spread)' (or also an '*$r-(sn,q)$*-spread').*

More generally,

Definition 7. *A partition of an* rsn*-dimensional vector space over* $GF(q)$ *by mutually disjoint* sn*-dimensional subspaces shall be called an '*r*-*(sn,q)*-spread'. In the literature, this is often called an '*sn*-spread' or projectively on the associated projective space as an '*$sn-1$*-spread'.*

If $r=2$*, any '*2*-*(sn,q)*-spread' is equivalent to a translation plane of order* q^{2n}*, with kernel containing* $GF(q)$*.*

Definition 8. *Let* Σ *be an (*r*–(*sn*-spread). We define the 'collineation group' of* Σ *to be the subgroup of* $\Gamma L(rsn,q)$ *that permutes the spread elements (henceforth called 'components').*

For a Desarguesian r*-spread* Σ*, the subgroup with elements*

$$(x_1,x_2,...,x_r)\longmapsto(dx_1,dx_2,...,d_rx_r)$$

for all d *nonzero in* $GF(q^{sn})$ *is called the '*sn*-kernel' subgroup* K_{sn}^* *of* Σ*. The group fixes each Desarguesian component and acts transitively on its points. The group* K_{sn}^* *union the zero mapping is isomorphic to* $GF(q^{sn})$*.* K_{sn}^* *has a subgroup* K_s^**, where* d *above is restricted to* $GF(q^s)^*$*, and* K_s^* *union the zero mapping is isomorphic to* $GF(q^s)$*.* K_s^* *is called the '*s*-kernel' subgroup'.*

Definition 9. *More generally, also for a Desarguesian spread Π, we note that the group $G^{(sn)^r}$ of order $(q^{sn}-1)^r$ with elements*

$$(x_1, x_2, ..., x_r) \longmapsto (d_1x_1, d_2x_2, ..., d_rx_r); d_i \in GF(q^{sn})^*, i = 1, 2, ..., r$$

also acts as a collineation group of Π. We call $G^{(sn)^r}$, the 'generalized kernel group'.

Let Σ be a Desarguesian r-spread with vectors $(x_1, x_2, ..., x_r)$. Consider any set Σ_j, and suppress the set of j zeros and write vectors in the form $(x_1^*, x_2^*, ..., x_{r-j}^*)$, in the order of non-zero elements within $(x_1, ..., x_r)$. Assume that $j \leq r-1$.

Definition 10. *We consider such vectors of the following form*

$$(x_1^*, x_1^{*q^{\lambda_1}}m_1, ..., x_1^{*q^{\lambda_{r-j}}}m_{r-j}).$$

If x_1^* varies over $GF(q^{sn})$, and consider Σ as an rsn-vector space over $GF(q)$, we then have an sn-vector subspace over $GF(q)$ that we call

$$y = (x_1^{*q^{\lambda_1}}m_1, ..., x_1^{*q^{\lambda_{r-j-1}}}m_{r-j-1}),$$

where λ_i are integers between 0 and $sn-1$.

Definition 11. *We are interested in the set of Desarguesian sn-subspaces*

$$y = (x_1^*w_1, ..., x_1^*w_{r-j-1})$$

(using the same notation) that can intersect

$$y = (x_1^{*q^{\lambda_1}}m_1, ..., x_1^{*q^{\lambda_{r-j-1}}}m_{r-j-1}).$$

We note that we have a non-zero intersection if and only if

$$x_1^{*q^{\lambda_i}-1} = w_i/m_i, \text{ for all } i = 1, 2, ..., r-j-1.$$

This set of non-zero intersections of Σ shall be called an 'extended André set of type $(\lambda_1, \lambda_2, ..., \lambda_{r-j-1})$'. The set of all subspaces

$$y = (x_1^{*q^{\lambda_1}}n_1, ..., x_1^{*q^{\lambda_{r-j-1}}}n_{r-j-1}),$$

such that

$$x_1^{*q^{\lambda_i}-1} = w_i/n_i, \text{ for all } i = 1, 2, ..., r-j-1,$$

has a solution is called an 'extended André replacement'.

Theorem 2. *(Johnson [11])Let Σ be a Desarguesian r-spread of order q^{sn}. Let Σ_j be the j-(0-set) for $j = 0, 1, 2, ..., r-1$.*

Choose any sn-dimensional subspace

$$y = (x_1^{*q^{\lambda_1}}n_1, ..., x_1^{*q^{\lambda_{r-j-1}}}n_{r-j-1});$$

where $n_i \in GF(q^{sn})^*$, $i = 1, 2, ..., r-j-1$. Let $d = (\lambda_1, \lambda_2, .., \lambda_{r-j-1})$, where $0 \leq \lambda_i \leq sn$.

(1) Then

$$A_{(n_1,\dots,n_{r-j-1})} = \left\{ \begin{array}{c} y = (x_1^* w_1, \dots, x_1^* w_{r-j-1}); \\ \text{there is an } x_1^* \text{ such that } x_1^{*q^{\lambda_i}-1} = w_i/n_i, \text{ for all } i = 1, 2, \dots, r-j-1 \end{array} \right\}$$

is a set of $(q^{sn}-1)/(q^d-1)$ *sn-subspaces, which is covered by the set of* $(q^{sn}-1)/(q^d-1)$

$$A_{(n_1,\dots,n_{r-j-1})}^{(\lambda_1,\dots,\lambda_{r-j-1})} = \left\{ y = (x_1^{*q^{\lambda_1}} n_1 d^{1-q^{\lambda_1}}, \dots, x_1^{*q^{\lambda_{r-j-1}}} n_{r-j-1} d^{1-q^{\lambda_{r-j-1}}}); d \in GF(q^{sn})^* \right\}.$$

(2) Let $C_{(q^{sn}-1)/(q-1)}$ *denote the cyclic subgroup of* $GF(q^{sn})^*$ *of order* $(q^{sn}-1)/(q-1)$*. Then, for each* $y = (x_1^* w_1, \dots, x_1^* w_{r-j-1})$*, there exists an element* τ *in* $C_{(q^{sn}-1)/(q-1)}$ *such that*

$$w_i = n_i \tau^{(q^{\lambda_i}-1)/(q-1)}.$$

(3) The $(q^{sn}-1)/(q^{(\lambda_1,\dots,\lambda_{r-j-1})}-1)$ *components of*

$$\left\{ y = (x_1^{*q^{\lambda_1}} n_1 d^{1-q^{\lambda_1}}, \dots, x_1^{*q^{\lambda_{r-j-1}}} n_{r-j-1} d^{1-q^{\lambda_{r-j-1}}}); d \in GF(q^{sn})^* \right\}$$

are in

$$\left((q^{sn}-1)/(q^{(\lambda_1,\dots,\lambda_{r-j-1})}-1)\right) / \left((q^s-1)/(q^{(\lambda_1,\lambda_2,\dots,\lambda_{r-j-1},s)}-1)\right)$$

orbits of length

$$(q^s-1)/(q^{(\lambda_1,\lambda_2,\dots,\lambda_{r-j-1},s)}-1)$$

under the s-kernel homology group K_s*.*

3 A Primer on Quasi-Subgeometries and Subgeometries

The difference between quasi-subgeometry partitions and subgeometry partitions depends simply on what one is calling a 'line'. It is possible that various constructions of quasi-subgeometry partitions are actually subgeometry partitions, but until now that question has been essentially unanswered. However, it is easy to find quasi-subgeometry partitions that are not subgeometry partitions. We include this section, which consists basically of two examples, as this will allow the reader a glimpse of the theory to come.

3.1 Quasi-Subgeometries

Suppose one would consider $PG(1,q^8)$ and ask if it would be possible to construct a quasi-subgeometry of $PG(1,q^8)$? A subgeometry is a set Γ of the points such that 'lines' in the substructure are simply intersections with Γ and lines of the projective superspace. So, certainly it is possible that we could have a $PG(1,q^2)$, for example, or a $PG(0,q^8)$, but since there is exactly one line, it would be impossible to have a $PG(t-1,q^j)$, for t not 2 or 1.

So, any non-trivial subgeometry partition must necessarily require that the subgeometries are $PG(0,q^8)$'s and $PG(1,q^k)$, for $k<8$.

However, it is possible to construct quasi-subgeometries, as follows. We have a 2-dimensional $GF(q^8)$ vector space V_2. Consider the 8-dimensional subspace over $GF((q),\left\{(x,x^{q^2});x\in GF(8)\right\}$. Call this subspace $y=x^{q^2}$ and consider the orbit of this subspace under the group

$$G_2=\left\langle (x,y)\longmapsto (xd,yd);d\in GF(q^8)^*\right\rangle .$$

We note that $y=x^{q^2}$ will map to $y=x^{q^2}d^{1-q^2}$, so there will be an orbit Γ of length $(q^8-1)/(q^2-1)$. We note that the point orbits that lie on Γ define 1-dimensional $GF(q^8)$-subspaces. Now define the set of 'points' to be the set of point-orbits under the group G that lie within Γ and define a 'line' to be the set of 'points' corresponding to a 2-dimensional vector subspace over $GF(q^2)$, generated by two distinct orbits P and Q. That is, a 'line' is the set of points that lie in $\{\alpha P+\beta Q;\alpha,\beta\in GF(q^2)\}$. Then it is not difficult to show that this set of points and lines is isomorphic to $PG(3,q^2)$. If we note that $\left\{y=x^{q^2}d^{1-q^2}\right\}=\left\{y=x\tau^{(q^2-1)/(q-1)};\tau\text{ has order dividing }(q^8-1)/(q-1)\right\}$, then we may replace this latter set by the former set. What we are actually doing is constructing an André plane from a Desarguesian plane by net replacement. If we consider $\{y=x^qnd^{1-q}\}$ $=\{y=x\tau n;\tau\text{ has order dividing }(q^8-1)/(q-1)\}$, then for appropriate n, the sets are disjoint on subspaces.

Hence, **In this way, it is possible to obtain a quasi-subgeometry partition of $PG(1,q^8)$ by quasi-subgeometries isomorphic to $PG(3,q^2)$, $PG(1,q^4)$, and $PG(0,q^8)$'s.**

3.2 Subgeometries

Now consider $PG(2,q^8)$ and ask again if there are subgeometry partitions of $PG(2,q^8)$, by subgeometries isomorphic to projective spaces other than $PG(2,q^k)$, $PG(1,q^t)$, or $PG(0,q^z)$? Now since there are $1+q^8+q^{16}$ lines, it is at least possible to construct such partitions.

So, consider a 3-dimensional $GF(8)$-vector space V_8 and consider the following subspace over $GF(q^2)$

$$\left\{(x,x^{q^2},x^{q^6});x\in GF(8)\right\}.$$

Call this subspace $y=(x^{q^2},x^{q^6})$. Consider the group

$$G_3=\left\langle (x,y,z)\longmapsto (xd,yd,zd);d\in GF(q^8)^*\right\rangle .$$

Then the image set Γ of this subspace under G_3 is

$$\left\{y=(x^{q^2}d^{1-q^2},x^{q^6}d^{1-q^6});d\in GF(q^8)^*\right\}$$

and has orbit length

$$(q^8-1)/(q^{(2,6,8)}-1)=(q^8-1)/(q^2-1).$$

We note that this set of 8-dimensional subspaces over $GF(q)$ is similarly covered by

$$\left\{y = (x\tau^{(q^2-1)/(q-1)}, x\tau^{(q^6-1)/(q-1)}); \tau \text{ has order dividing } (q^8-1)/(q-1)\right\}.$$

So we claim that in this setting, the orbit forms a 'subgeometry' of $PG(2, q^8)$ that is isomorphic to $PG(3, q^2)$. The 'points' of the point-line incidence structure S_Γ are the point orbits under $GF(q^8)$, that lie within Γ. The 'lines' of S_Γ are the intersections in Γ with lines of $PG(2, q^8)$, that is the intersections with Γ of 2-dimensional vector spaces over $GF(q^8)$, generated by two distinct 1-dimensional $GF(q^8)$, subspaces that lie within Γ. We note that there are $(q^8-1)/(q^2-1)$ 'points' so if we were to obtain a projective space, it would be isomorphic to $PG(3, q^2)$.

So, take two distinct orbits P and Q under G_3 that lie within Γ. These become 1-dimensional $GF(q^8)$-spaces. Take a generator from each and form the 2-dimensional $GF(q^8)$-space. Since in P and Q there are always generators in $y = (x^{q^2}, x^{q^6})$, suppose

$$\left\langle (x_1, x_1^{q^2}, x_1^{q^6}) \right\rangle = P \text{ and } \left\langle (x_2, x_2^{q^2}, x_2^{q^6}) \right\rangle = Q.$$

Let $\alpha, \beta \in GF(q^8)$ and generate the two-dimensional vector space

$$\left\langle (x_1, x_1^{q^2}, x_1^{q^6}), (x_2, x_2^{q^2}, x_2^{q^6}) \right\rangle = \left\{\alpha(x_1, x_1^{q^2}, x_1^{q^6}) + \beta(x_2, x_2^{q^2}, x_2^{q^6}); \alpha, \beta \in GF(q^8)\right\}.$$

Now form

$$\left\{\alpha(x_1, x_1^{q^2}, x_1^{q^6}) + \beta(x_2, x_2^{q^2}, x_2^{q^6}); \alpha, \beta \in GF(q^8)\right\} \cap \Gamma$$

and let R be an orbit of Γ different from P and Q such that

$$R = \left\langle \alpha(x_1, x_1^{q^2}, x_1^{q^6}) + \beta(x_2, x_2^{q^2}, x_2^{q^6}) \right\rangle,$$

for some fixed α, β, now both non-zero. Again, R contains a vector of $y = (x^{q^2}, q^{q^6})$ that generates R over $GF(q^8)$. Let

$$R = \left\langle (x_3, x_3^{q^2}, x_3^{q^6}) \right\rangle.$$

Then it follows that there is a non-zero element of $GF(q^8)$, so that

$$\rho(\alpha(x_1, x_1^{q^2}, x_1^{q^6}) + \beta(x_2, x_2^{q^2}, x_2^{q^6})) = (x_3, x_3^{q^2}, x_3^{q^6}).$$

Hence, ρ, α, β are all non-zero and since scalar addition by non-zero elements of $GF(q^8)$, is given by the group G_3, we may incorporate ρ into α and β, and assume without loss of generality that

$$(\alpha(x_1, x_1^{q^2}, x_1^{q^6}) + \beta(x_2, x_2^{q^2}, x_2^{q^6})) = (\alpha x_1 + \beta x_2, \alpha x_1^{q^2} + \beta x_2^{q^2}, \alpha x_1^{q^6} + \beta x_2^{q}) = (x_3, x_3^{q^2}, x_3^{q^6}).$$

Hence,

$$\alpha x_1 + \beta x_2 = x_3,\ \alpha x_1^{q^2} + \beta x_2^{q^2} = x_3^{q^2} \text{ and } \alpha x_1^{q^6} + \beta x_2^{q} = x_3^{q^6}.$$

This leads to the following two equations:

$$(*) : (\alpha - \alpha^{q^2}) x_1^{q^2} + (\beta - \beta^{q^2}) x_2^{q^2} = 0$$

and

$$(**) : (\alpha - \alpha^{q^6})x_1^{q^6} + (\beta - \beta^{q^2})x_2^{q^6} = 0.$$

Since x_1, x_2, x_3 are all non-zero, then $\alpha - \alpha^{q^2} = 0$ if and only if $\beta - \beta^{q^2} = 0$, which is valid if and only if α and β are in $GF(q^2)$. Similarly, $\alpha - \alpha^{q^6} = 0$ if and only if $\alpha^{q^6} = \alpha$, so that α has order dividing $(q^6 - 1, q^8 - 1) = (q^2 - 1)$, so again, α and β are in $GF(q^2)$. Hence, we may assume that $\alpha - \alpha^{q^2}$ and $\alpha - \alpha^{q^6}$ are both non-zero. So, we obtain

$$(*)' \ : \ \left(\frac{x_1}{x_2}\right)^{q^2} = \left(\frac{\beta^{q^2} - \beta}{\alpha - \alpha^{q^2}}\right), \text{ and}$$
$$(**)' \ : \ \left(\frac{x_1}{x_2}\right)^{q^6} = \left(\frac{\beta^{q^6} - \beta}{\alpha - \alpha^{q^6}}\right).$$

Raise $(*)'$ to the q^6-th power to obtain:

$$(*)' : \left(\frac{x_1}{x_2}\right)^{q^8} = \left(\frac{\beta^{q^8} - \beta^{q^6}}{\alpha^{q^6} - \alpha^{q^8}}\right) = \left(\frac{\beta - \beta^{q^6}}{\alpha^{q^6} - \alpha}\right) = \left(\frac{\beta^{q^6} - \beta}{\alpha - \alpha^{q^6}}\right).$$

Therefore, we arrive at the equation:

$$\left(\frac{x_1}{x_2}\right)^{q^6} = \left(\frac{x_1}{x_2}\right)^{q^8}.$$

This is equivalent to

$$\left(\frac{x_1}{x_2}\right)^{q^8 - q^6} = \left(\frac{x_1}{x_2}\right)^{q^6(q^2-1)} = 1.$$

Therefore, $\left(\frac{x_1}{x_2}\right)^{q^6}$ is in $GF(q^2)^*$, which implies that $\left(\frac{x_1}{x_2}\right)$ is in $GF(q^2)$. Let $x_1 = x_2\delta$, for $\delta \in GF(q^2)^*$ then $(x_1, x_1^{q^2}, x_1^{q^6}) = (\delta x_2, \delta x_2^{q^2}, \delta x_2^{q^6})$, since $\delta^{q^2} = \delta$. However, this means that $P = Q$, originally.

Hence, it can only be that $\alpha, \beta \in GF(q^2)^*$. Therefore, there are $(q^2)^2 + 1$ 'points' of intersection with a line of $PG(2, q^8)$ and S_Γ. It follows directly that S_Γ is isomorphic to $PG(3, q^2)$.

In a similarly manner we may obtain subgeometries isomorphic to $PG(1, q^4)$ and $PG(0, q^8)$. **Therefore, we obtain a subgeometry partition of $PG(2, q^8)$, by subgeometries isomorphic to $PG(3, q^2)$, $PG(1, q^4)$ and $PG(0, q^8)$.**

If it is objectionable to include 'points' within a partition then simply take $PG(3n - 1, q^8)$, for $n > 1$, and it is similarly possible to construct subgeometry partitions by subgeometries isomorphic to $PG(4n - 1, q^2)$, $PG(2n - 1, q^4)$ and $PG(n - 1, q^8)$.

In the following, we shall basically repeat the above construction in r-dimensional vector spaces over $GF(q^{sn})$ and construct various subgeometries isomorphic to $PG(sn/s^* - 1, q^{s^*})$, where s^* is any divisor of s. It is possible to obtain subgeometry partitions using a wide variety of different isomorphism types of projective spaces as subgeometry partitions.

4 Subgeometries

Let Σ again be a Desarguesian r-spread of order q^{sn}. We may regard Σ as an rsn-vector space over $GF(q)$. Moreover, since the kernel homology group K_s determines a field $K_s \cup \{0\} = K_s^+$, we may also regard Σ as an rn-vector space over K_s^+: Letting

$$\tau_d : (x_1, x_2, ..., x_r) \longmapsto (x_1 d, x_2 d, ..., x_r d); d \in GF(q^s), \text{ with } \tau_0, \text{ the zero mapping,}$$

then $K_s^+ = \langle \tau_d; d \in GF(q^s) \rangle$. Define $\tau_d v = d \odot v = (x_1 d, x_2 d, ..., x_r d)$, for $v = (x_1, x_2, ..., x_r)$, then clearly Σ is an rn-dimensional K_s^+-space. Let the lattice of subspaces be denoted by $PG(rn-1, q^s)$. Then clearly, Σ is an rn-dimensional subspace over K_s^+. The lattice of vector subspaces then forms a projective space $PG(rn-1, q^s)$.

4.1 The Main Theorem on Subgeometries

Theorem 3. *Let the lattice of vector subspaces of Σ over K_s^+ be denoted by $PG(rn-1, q^s)$. Take any*

$$\begin{aligned} & A_{(n_1,...,n_{r-j-1})}^{(\lambda_1,...,\lambda_{r-j-1})} \\ = \; & \left\{ y = (x_1^{*q^{\lambda_1}} n_1 d^{1-q^{\lambda_1}}, ..., x_1^{*q^{\lambda_{r-j-1}}} n_{r-j-1} d^{1-q^{\lambda_{r-j-1}}}); d \in GF(q^{sn})^* \right\}. \end{aligned}$$

Again, there are

$$\Big((q^{sn}-1)/(q^{(\lambda_1,...,\lambda_{r-j-1})}-1)\Big) \,/\, \Big((q^s-1)/(q^{(\lambda_1,\lambda_2,...,\lambda_{r-j-1},s)}-1)\Big)$$

component orbits of length

$$(q^s-1)/(q^{(\lambda_1,\lambda_2,...,\lambda_{r-j-1},s)}-1)$$

under K_s.

Let $s^* = (\lambda_1, \lambda_2, ..., \lambda_{r-j-s})$. Assume that within $(\lambda_1, \lambda_2, ..., \lambda_{r-j-1})$, there are elements λ_l and λ_k such that $\lambda_l = s^*\rho$ and $\lambda_k = s^*\rho(s-1)$. If $s = 2$, λ_l and λ_k are equal and it is then possible that $r-j-1 = 1$. Each such orbit becomes a projective subgeometry of $PG(rn-1, q^s)$, isomorphic to $PG(sn/(\lambda_1, \lambda_2, ..., \lambda_{r-j-1}, s) - 1, q^{(\lambda_1,\lambda_2,...,\lambda_{r-j-1},s)})$, for $r-j-1 \geq 2$, provided $s > 2$ and $r-j-1 \geq 1$ for $s = 2$.

Proof. From Johnson [10] (see the background section), all parts are proved with the exception of proving that the constructed quasi-subgeometry is actually a subgeometry. Let

$$L : y = (x^{q^{\lambda_1}} m_1, x^{q^{\lambda_2}} m_2 ...)$$

be in the K_s-orbit. Choose two vectors v_1 and v_2 that lie in the union of the subspaces in the orbit and form $\langle v_1, v_2 \rangle_{K_s^+}$, the 2-dimensional K_s^+-subspace, assuming that $\{v_1, v_2\}$ is linearly independent over K_s^+. Since each 1-dimensional K_s^+-subspace non-trivially intersects L, we may assume that v_1 and v_2 lie on L. Hence, let

$$v_i = (x_i, x_i^{q^{\lambda_1}} m_1, x_i^{q^{\lambda_2}} m_2 ...), \text{ for } i = 1, 2.$$

To show that we obtain a sub-geometry, we need to show that the intersection of $\langle v_1, v_2\rangle_{K_s^+}$, with 1-dimensional K_s^+-subspaces that lie in the orbit of L may be given via scalars in $GF(q^{(\lambda_1,\ldots,\lambda_{r-j-1},s)}) = GF(q^{s^*})$. Any intersection of $\langle v_1, v_2\rangle_{K_s^+}$ contains a vector on L and in that particular 1-dimensional K_s^+-subspace. Hence, assume that

$$\alpha v_1 + \beta v_2 = v_3,$$

where v_3 is also on L. We need to show that α and β are in $GF(q^{s^*})$.

So, let

$$v_i = (x_i, x_i^{q^{\lambda_1}} m_1, x_i^{q^{\lambda_2}} m_2 ...), \text{ for } i = 1, 2, 3.$$

Then

$$\alpha(x_1, x_1^{q^{\lambda_1}} m_1, x_1^{q^{\lambda_2}} m_2 ...) + \beta(x_2, x_2^{q^{\lambda_1}} m_1, x_2^{q^{\lambda_2}} m_2 ...) = (x_3, x_3^{q^{\lambda_1}} m_1, x_3^{q^{\lambda_2}} m_2 ...)$$

if and only if

$$(\alpha x_1 + \beta x_2)^{q^{\lambda_i}} m_i = \alpha x_1^{q^{\lambda_i}} m_i + \beta x_1^{q^{\lambda_i}} m_i, \text{ for all } i = 1, 2, ..., r-j-1.$$

This leads to the following equivalent set of equations:

$$x_1^{q^{\lambda_i}}(\alpha - \alpha^{q^{\lambda_i}}) + x_2^{q^{\lambda_i}}(\beta - \beta^{q^{\lambda_i}}) = 0, \text{ for all } i = 1, 2, ..., r-j-1.$$

We now restrict to the two equations (or one if $s = 2$) for $i = l$ and k :

$$(*) : x_1^{q^{s^*\rho}}(\alpha - \alpha^{q^{s^*\rho}}) + x_2^{q^{s^*\rho}}(\beta - \beta^{q^{s^*}}) = 0,$$

$$(**) : x_1^{q^{s^*\rho(s-1)}}(\alpha - \alpha^{q^{s^*\rho(s-1))}}) + x_2^{q^{s^*\rho(S-1)}}(\beta - \beta^{s^*\rho(s-1))}) = 0.$$

Note that $\alpha, \beta \in GF(q^{s^*})$ if and only if $\alpha - \alpha^{q^{s^*\rho}} = \beta - \beta^{q^{s^*\rho}} = 0$. Moreover, the previous two equations show that $\alpha - \alpha^{q^{s^*\rho}} = 0$ if and only if $\beta - \beta^{q^{s^*\rho}} = 0$. Since this is what we would like to prove, assume otherwise, that $\alpha - \alpha^{q^{s^*}} \neq 0$. Then,

$$(*)' : \left(\frac{x_1}{x_2}\right)^{q^{s^*\rho}} = \frac{\beta^{q^{s^*\rho}} - \beta}{\alpha - \alpha^{q^{s^*\rho}}}$$

and

$$(**)' : \left(\frac{x_1}{x_2}\right)^{q^{s^*(s-1)}} = \frac{\beta^{q^{s^*(s-1)}} - \beta}{\alpha - \alpha^{q^{s^*(s-1)}}}.$$

From $(*)'$, we obtain

$$\begin{aligned}\left(\left(\frac{x_1}{x_2}\right)^{q^{s^*\rho}}\right)^{q^{s^*\rho(s-1)}} &= \left(\frac{\beta^{q^{s^*\rho}} - \beta}{\alpha - \alpha^{q^{s^*\rho}}}\right)^{q^{s^*\rho(s-1)}} = \left(\frac{\beta^{q^{s^*\rho s}} - \beta^{q^{s^*\rho(s-1)}}}{\alpha^{q^{s^*\rho(s-1)}} - \alpha^{q^{s^*\rho s}}}\right) \\ &= \left(\frac{\beta - \beta^{q^{s^*\rho(s-1)}}}{\alpha^{q^{s^*\rho(s-1)}} - \alpha}\right) = \left(\frac{\beta^{q^{s^*\rho(s-1)}} - \beta}{\alpha - \alpha^{q^{s^*\rho(s-1)}}}\right).\end{aligned}$$

That is,

$$(***)' : \left(\frac{x_1}{x_2}\right)^{q^{s^*\rho s}} = \left(\frac{\beta^{q^{s^*\rho(s-1)}} - \beta}{\alpha - \alpha^{q^{s^*\rho(s-1)}}}\right).$$

Using, $(**)'$, we have

$$\left(\frac{x_1}{x_2}\right)^{q^{s^*\rho s}} = \left(\frac{x_1}{x_2}\right)^{q^{s^*\rho(s-1)}},$$

or equivalently,

$$\left(\frac{x_1}{x_2}\right)^{q^{s^*\rho s s^*\rho} - q^{s^*\rho(s-1)}} = 1 = \left(\frac{x_1}{x_2}\right)^{q^{s^*\rho(s-1)}(q^{s^*\rho}-1)}.$$

But, this says that

$$\left(\frac{x_1}{x_2}\right)^{q^{s^*\rho(s-1)}} \in GF(q^{s^*}),$$

which, in turn, implies that

$$\left(\frac{x_1}{x_2}\right) \in GF(q^{s^*}).$$

But we know that

$$(*)' : \left(\frac{x_1}{x_2}\right) = \left(\frac{x_1}{x_2}\right)^{q^{s^*\rho}} = \frac{\beta^{q^{s^*\rho}} - \beta}{\alpha - \alpha^{q^{s^*\rho}}},$$

and hence

$$\left(\frac{x_1}{x_2}\right)^{q^{\lambda_i}} = \frac{\beta^{q^{s^*\rho}} - \beta}{\alpha - \alpha^{q^{s^*\rho}}}, \text{ for all } i,$$

which is equivalent to

$$(*)^+ : x_1^{q^{\lambda_i}}(\alpha - \alpha^{q^{s^*\rho}}) + x_2^{q^{\lambda_i}}(\beta - \beta^{q^{s^*\rho}}) = 0, \text{ for all } i = 0, 1, ..., r - j - 1.$$

Now consider

$$(\alpha - \alpha^{q^{s^*\rho}})(x_1, x_1^{q^{\lambda_1}} m_1, x_1^{q^{\lambda_2}} m_2...) + (\beta - \beta^{q^{s^*\rho}})(x_2, x_2^{q^{\lambda_1}} m_1, x_2^{q^{\lambda_2}} m_2...)$$

and note that we obtain 0, in each coordinate: Hence,

$$\begin{aligned} & (\alpha - \alpha^{q^{s^*\rho}})(x_1, x_1^{q^{\lambda_1}} m_1, x_1^{q^{\lambda_2}} m_2...) + (\beta - \beta^{q^{s^*\rho}})(x_2, x_2^{q^{\lambda_1}} m_1, x_2^{q^{\lambda_2}} m_2...) \\ = \; & (0, 0, 0, ...0). \end{aligned}$$

Since the two vectors are linearly independent over $GF(q^s)$, it follows that

$$(\alpha - \alpha^{q^{s^*\rho}}) = 0 = (\beta - \beta^{q^{s^*\rho}}),$$

a contradiction and hence the proof that we obtain a subgeometry in this situation. ∎

4.2 The Main Theorem on Subgeometry Partitions

In the previous subsection, we isolated on the construction of subgeometries. We now show how to find subgeometry partitions.

Theorem 4. *Let D_s denote the set of divisors of s (including 1 and s). When a replacement set of $(q^{sn}-1)/(q^{s^*}-1)$ sn-spaces is obtained for $s^* \in D_s$, let k_{s^*} denote the number of different and mutually disjoint replacement sets of $(q^{sn}-1)/(q^{s^*}-1)$ sn-spaces (k_{s^*} could be 0). Then we merely require that*

$$\sum_{s^* \in D_s} (\frac{q^{sn}-1}{q^{s^*}-1})k_{s^*} = (q^{sn}-1)^{r-j-1}.$$

Now we can actually do this for each of the $\binom{r}{r-j}$ $j-$ (0-subsets). Let $k_{s^*,j,w}$ be the number of different and mutually disjoint replacement sets of $(q^{sn}-1)/(q^{s^*}-1)$ sn-spaces in $\Sigma_{j,w}$. Then, considering the $j-$ (0-sets), for $r-j \geq 3$, we require

$$\sum_{j=0}^{r-3} \sum_{w=1}^{\binom{r}{r-j}} \sum_{s^* \in D_s} (\frac{q^{sn}-1}{q^{d^*}-1})k_{d^*} = (q^{rsn}-1)/(q^{sn}-1) - \binom{r}{r-2}(q^{sn}-1) - \binom{r}{r-1}.$$

Let $s^* = \gcd(\lambda_1, \lambda_2, ..., \lambda_{r-j-1}, s)$. Assume that within $(\lambda_1, \lambda_2, ..., \lambda_{r-j-1})$, there are elements λ_l and λ_k such that $\lambda_l = s^*\rho$ and $\lambda_k = s^*\rho(s-1)$. If $s = 2$, λ_l and λ_k are equal and it is then possible that $r-j-1 = 1$. Then each orbit under the kernel group of order $q^s - 1$ becomes a projective subgeometry of $PG(rn-1, q^s)$, isomorphic to $PG(sn/(\lambda_1, \lambda_2, ..., \lambda_{r-j-1}, s) - 1, q^{(\lambda_1,\lambda_2,...,\lambda_{r-j-1},s)})$, for $r-j-1 \geq 2$, provided $s > 2$ and $r-j-1 \geq 1$ for $s = 2$.

Therefore, the associated $PG(rn-1, q^s)$ is partitioned by subgeometries as follows: There are

$$\sum_{j=0}^{r-1} \sum_{w=1}^{\binom{r}{r-j}} k_{s^*,j,w}$$

subgeometries isomorphic to $PG(sn/s^*-1, q^{s^*})$ (the $j = r-1$ sets always lead to $\binom{r}{r-1} = r$ $PG(n-1, q^s)$'s).

The reader is also directed to Johnson [11] for an algorithmic approach to the selection of subgeometries within the partition.

5 The Ebert-Mellinger Subgeometry Partitions

In [4], Ebert and Mellinger construct a class of subgeometry partitions of $PG(rn-1, q^r)$ by subgeometries isomorphic to $PG(rn-1, q)$ and $PG(n-1, q^r)$. Furthermore, it is shown that there is a class of $r-(rn, q)$-spreads constructed by 'geometric lifting'–what we are calling 'unfolding the fan'. We recall a theorem from the background section. That is, an associated spread in $PG(r^2n-1, q)$ (what Ebert and Mellinger term a $(rn-1)$-spread (in the projective sense). We recall a theorem of Johnson [11] on the Ebert-Mellinger $r-(rn, q)$-spreads.

5.1 The Ebert-Mellinger $r-(rn,q)$-Spreads

Ebert and Mellinger [4], construct a new $r-(rn,q)$-spread admitting an Abelian group of order $(q^{rn}-1)^2$ that may be constructed with the methods of the previous theorem. The construction in Ebert and Mellinger begins with the construction of new subgeometry partitions in $PG(rn-1,q^r)$ by subgeometries isomorphic to $PG(rn-1,q)$ and $PG(n-1,q^r)$.

Theorem 5. *(Johnson [11]) The $r-(rn,q)$-spreads of Ebert and Mellinger are extended André spreads. When $r=2$, the spreads correspond to André planes of order q^{2n}.*

So, there is a group of order q^r-1 that arises from the Ebert-Mellinger subgeometry partitions. Unfolding the fan(s), shows that the $PG(rn-1,q)$'s unfold to an orbit of rn-dimensional vector subspaces over $GF(q)$ of length $(q^r-1)/(q-1)$ and the $PG(n-1,q^r)$'s unfold to an rn-dimensional vector subspace over G. Now begin with a Desarguesian $r-(rn,q)$-spread and construct the generalized André type covers,

$$\begin{aligned} & A^{(\lambda_1,\ldots,\lambda_{r-j-1})}_{(n_1,\ldots,n_{r-j-1})} \\ = \; & \left\{ y=(x_1^{*q^{\lambda_1}}n_1 d^{1-q^{\lambda_1}},\ldots,x_1^{*q^{\lambda_{r-j-1}}}n_{r-j-1}d^{1-q^{\lambda_{r-j-1}}}); d\in GF(q^{sn})^* \right\}, \end{aligned}$$

where we only use the $j=0$-(0-sets), where we take $\lambda_1=1,\lambda_2=2,\ldots,\lambda_{r-1}=r-1$.

Therefore, the replacement sets 'all' have the general form:

$$\begin{aligned} & A^{(1,2,\ldots,r-1)}_{(n_1,\ldots,n_{r-1})} \\ = \; & \left\{ y=(x^q n_1 d^{1-q},\ldots,x^{q^{r-1}}n_{r-11}d^{1-q^{r-j-1}}); d\in GF(q^{rn})^* \right\}. \end{aligned}$$

Furthermore, the other $j-$(0-sets) are not replaced. Specially, we recall our main theorem on subgeometries.

Theorem 6. *In this setting, we have $s=r$ and $j=0$, $\lambda_i=i$, so*

$$(q^s-1)/(q^{(\lambda_1,\lambda_2,\ldots,\lambda_{r-j-1},s)}-1)=(q^r-1)/(q-1).$$

So, we obtain a subgeometry partition of $PG(rn-1,q^r)$ by $PG(rn-1,q)$'s and $PG(n-1,q^r)'s$. These are the subgeometry partitions constructed by Ebert and Mellinger. This also means that since the group arises from a Desarguesian $r-(rn,q)$-spread, all remaining components of the spread are fixed by the group.

The reader is also directed back to Theorem 2 for more of the details. But, note that by making different replacements of the indicated generalized André spreads and also make replacements in other $j-$(0-sets), a great variety of subgeometry partitions may be obtained, with varying and many different types of subgeometries.

6 Final Comments

All of the material presented in the article may be generalized to the infinite case in a manner considered in the author's work [10]. Furthermore, in the finite case, there are also different replacements of the generalized André nets, that are not André replacements and similar to work given in Johnson [9] for the translation plane case. This work will be presented in a forth-coming article.

In general, we have constructed subgeometry partitions of $PG(rn-1,q^s)$ by a set of subgeometries isomorphic to $PG(sn/s^*-1,q^{s^*})$ for any subset of the divisors of s. So, the number of mutually non-isomorphic subgeometries in the partitions simply depends on the number of divisors of s. For example, if $s=2\cdot 3\cdot 5$, there are subgeometries possible of any subset of the following set

$$\left\{ \begin{array}{c} PG(30n-1,q), PG(15n-1,q^2), PG(10n-1,q^3), \\ PG(6n-1,q^5), PG(5n-1,q^6), PG(3n-1,q^{10}), \\ PG(2n-1,q^{15}), PG(n-1,q^{30}) \end{array} \right\}.$$

From our constructions, we finally note the following theorem.

Theorem 7. *Given any projective space* Π *isomorphic to* $PG(rn-1,q^s)$*, for* $r\geq 3$*, and any integers* n *and* s*. Let* D_s *denote the set of divisors of* s*, including* 1 *and* s*. Then there exist subgeometry partitions of* Π *by subgeometries* $PG(sn/s^*-1,q^{s^*})$*, for all* $s^*\in D_s$*.*

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