Polynomial characterization of Asplund spaces

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Abstract

We prove that, given an index m, if every Pietsch integral m-homogeneous polynomial on a Banach space E is nuclear, then E is Asplund. The converse was proved by Alencar.

A Banach space E is *Asplund* if every separable subspace of E has a separable dual, equivalently, if the dual of E has the Radon-Nikodým property. A short introduction to Asplund spaces may be seen in [DGZ, I.5] and a more detailed one is contained in [Y].

For Banach spaces E and F, we use the notation $\mathcal{L}_{\mathrm{PI}}(E,F)$ for the space of all Pietsch integral operators from E into F, and $\mathcal{L}_{\mathrm{N}}(E,F)$ for the nuclear operators (see definitions in [DU]). The following result is proved in [A1]:

Theorem 1. *A Banach space E is Asplund if and only if, for every Banach space F, we have $\mathcal{L}_{\mathrm{PI}}(E,F) = \mathcal{L}_{\mathrm{N}}(E,F)$.*

Given an integer m, we use the notation $\mathcal{P}_{\mathrm{PI}}(^mE,F)$ for the space of m-homogeneous Pietsch integral polynomials from E into F, and $\mathcal{P}_{\mathrm{N}}(^mE,F)$ for the nuclear polynomials (see definitions below). It is proved in [A2] that, if E is Asplund, and m is an integer, then $\mathcal{P}_{\mathrm{PI}}(^mE,F) = \mathcal{P}_{\mathrm{N}}(^mE,F)$ for every Banach space F.

Here we give a converse to this result, proving that the equality $\mathcal{P}_{\mathrm{PI}}(^mE,F) = \mathcal{P}_{\mathrm{N}}(^mE,F)$ for some m implies $\mathcal{L}_{\mathrm{PI}}(E,F) = \mathcal{L}_{\mathrm{N}}(E,F)$. As a consequence, if every Pietsch integral polynomial on E is nuclear, then E is Asplund.

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To obtain this, we need to show that every 1-dominated polynomial on a $C(K)$ space is Pietsch integral, which extends a well-known result for linear operators.

We also prove that, if $T \in \mathcal{L}_{\mathrm{PI}}(E, F)$, then the operator

$$\otimes^m T : \otimes_\epsilon^m E \longrightarrow \otimes_\pi^m F$$

is well-defined and Pietsch integral.

Throughout, E and F denote Banach spaces, E^* is the dual of E, and B_E stands for its closed unit ball. By $\mathbb{N}$ we represent the set of all natural numbers. By an operator we always mean a linear bounded mapping between Banach spaces. The notation $\mathcal{L}(E, F)$ stands for the space of all operators from E into F. By $E \equiv F$, we mean that E and F are isometrically isomorphic. Given $m \in \mathbb{N}$, we denote by $\mathcal{P}(^m E, F)$ the space of all m-homogeneous (continuous) polynomials from E into F. Recall that with each $P \in \mathcal{P}(^m E, F)$ we can associate a unique symmetric m-linear $\widehat{P} : E \times \overset{(m)}{\cdots} \times E \to F$ so that

$$P(x) = \widehat{P}\left(x, \overset{(m)}{\ldots}, x\right) \qquad (x \in E).$$

For the general theory of polynomials on Banach spaces, we refer to [D] and [Mu].

We use the notation $\otimes^m E := E \otimes \overset{(m)}{\cdots} \otimes E$ for the m-fold tensor product of E, $\otimes_\epsilon^m E := E \otimes_\epsilon \overset{(m)}{\cdots} \otimes_\epsilon E$ for the m-fold injective tensor product of E, and $\otimes_\pi^m E$ for the m-fold projective tensor product of E (see [DU] for the theory of tensor products). By $\otimes_s^m E := E \otimes_s \overset{(m)}{\cdots} \otimes_s E$ we denote the m-fold symmetric tensor product of E, i.e., the set of all elements $u \in \otimes^m E$ of the form

$$u = \sum_{j=1}^n \lambda_j x_j \otimes \overset{(m)}{\cdots} \otimes x_j \qquad (n \in \mathbb{N}, \lambda_j \in \mathbb{K}, x_j \in E, 1 \leq j \leq n).$$

By $\otimes_{\pi,s}^m E$ we denote the closure of $\otimes_s^m E$ in $\otimes_\pi^m E$. For symmetric tensor products, we refer to [F]. For simplicity, we write $\otimes^m x := x \otimes \overset{(m)}{\cdots} \otimes x$. For $T \in \mathcal{L}(E, F)$, $\otimes^m T$ stands for the m-fold tensor product:

$$\otimes^m T := T \otimes \overset{(m)}{\cdots} \otimes T : \otimes^m E \longrightarrow \otimes^m F.$$

Given $P \in \mathcal{P}(^m E, F)$, let

$$\overline{\widehat{P}} : \otimes^m E \longrightarrow F$$

be the linearization of $\widehat{P}$, defined by

$$\overline{\widehat{P}} \left(\sum_{j=1}^n x_{1j} \otimes \cdots \otimes x_{mj} \right) = \sum_{j=1}^n \widehat{P}(x_{1j}, \ldots, x_{mj})$$

where $x_{kj} \in E$ $(1 \leq k \leq m, 1 \leq j \leq n)$.

A polynomial $P \in \mathcal{P}(^m E, F)$ is *nuclear* [D, Definition 2.9] if it can be written in the form

$$P(x) = \sum_{i=1}^\infty \left[x_i^*(x)\right]^m y_i \qquad (x \in E)$$

where $(x_i^*) \subset E^*$ and $(y_i) \subset F$ are sequences such that

$$\sum_{i=1}^{\infty} \|x_i^*\|^m \|y_i\| < \infty.$$

A polynomial $P \in \mathcal{P}(^mE, F)$ is *Pietsch integral* if it can be written in the form

$$P(x) = \int_{B_{E^*}} [x^*(x)]^m \, d\mathcal{G}(x^*) \qquad (x \in E)$$

where $\mathcal{G}$ is an F-valued regular countably additive Borel measure, of bounded variation, defined on B_{E^*}, where B_{E^*} is endowed with the weak-star topology. A similar definition may be given for the Pietsch integral multilinear mappings (see [A2]). Every nuclear polynomial is Pietsch integral.

Given $1 \le r < \infty$, a polynomial $P \in \mathcal{P}(^mE, F)$ is *r-dominated* (see, e.g., [M, MT]) if there exists a constant $k > 0$ such that, for all $n \in \mathbb{N}$ and $(x_i)_{i=1}^n \subset E$, we have

$$\left(\sum_{i=1}^{n} \|P(x_i)\|^{\frac{r}{m}}\right)^{\frac{m}{r}} \le k \sup_{x^* \in B_{E^*}} \left(\sum_{i=1}^{n} |x^*(x_i)|^r\right)^{\frac{m}{r}}.$$

For $m = 1$ we obtain the absolutely r-summing operators.

Proposition 2. *Let E, F, X, and Y be Banach spaces. Suppose that $P \in \mathcal{P}(^mE, F)$ is Pietsch integral, and let $T \in \mathcal{L}(X, E)$ and $S \in \mathcal{L}(F, Y)$. Then $S \circ P \circ T \in \mathcal{P}(^mX, Y)$ is Pietsch integral.*

Proof. It is enough to show that both $P \circ T$ and $S \circ P$ are Pietsch integral. If P is Pietsch integral, so is $\widehat{P}$, by [A2, Proposition 2]. This implies that the linearization

$$\overline{\widehat{P}} : \otimes_\epsilon^m E \longrightarrow F$$

is well-defined and Pietsch integral [V, Proposition 2.6]. Since $\otimes^m T : \otimes_\epsilon^m X \to \otimes_\epsilon^m E$ is continuous, we have that

$$\overline{\widehat{P}} \circ (\otimes^m T) : \otimes_\epsilon^m X \longrightarrow F$$

is Pietsch integral. Using the polarization formula [Mu, Theorem 1.10], we easily have

$$\overline{\widehat{P}} \circ (\otimes^m T) = \overline{\widehat{P \circ T}}.$$

Hence, $\overline{\widehat{P \circ T}}$ is Pietsch integral. By [V, Proposition 2.6], so is $\widehat{P \circ T}$ and, by [A2, Proposition 2], so is $P \circ T$.

Using the fact that

$$\overline{\widehat{S \circ P}} = S \circ \overline{\widehat{P}},$$

a similar argument shows that $S \circ P$ is Pietsch integral. ∎

The following result may be of independent interest:

Lemma 3. *Let $T \in \mathcal{L}_{\mathrm{PI}}(E, F)$ and $m \in \mathbb{N}$. Then the tensor product operator*

$$\otimes^m T : \otimes_\epsilon^m E \longrightarrow \otimes_\pi^m F$$

is well-defined and Pietsch integral.

Proof. By [DJT, Theorem 5.6], we can find a compact Hausdorff space K, a regular Borel probability measure μ on K, and operators $a : E \to C(K)$ and $b : L_1(K,\mu) \to F$ such that the following diagram commutes

$$\begin{tikzcd} E \arrow[r, "T"] \arrow[d, "a"'] & F \\ C(K) \arrow[r, "j"'] & L_1(K,\mu) \arrow[u, "b"'] \end{tikzcd}$$

where j is the natural inclusion.

The tensor product operator

$$\otimes^m j : \otimes_\epsilon^m C(K) \equiv C\left(K \times \overset{(m)}{\cdots} \times K\right) \to \otimes_\pi^m L_1(K,\mu) \equiv L_1\left(K \times \overset{(m)}{\cdots} \times K, \mu \times \overset{(m)}{\cdots} \times \mu\right)$$

is the natural inclusion, so it is Pietsch integral (see the definition in [DJT, page 97]). Hence, the composition

$$\otimes^m T : \otimes_\epsilon^m E \xrightarrow{\otimes^m a} \otimes_\epsilon^m C(K) \xrightarrow{\otimes^m j} \otimes_\pi^m L_1(K,\mu) \xrightarrow{\otimes^m b} \otimes_\pi^m F$$

is Pietsch integral. ∎

Proposition 4. *Let E, F, and G be Banach spaces. Assume that $T \in \mathcal{L}_{\mathrm{PI}}(E,F)$, and let $Q \in \mathcal{P}(^m F, G)$ be a polynomial. Then $P := Q \circ T \in \mathcal{P}(^m E, G)$ is Pietsch integral.*

Proof. By Lemma 3, the operator

$$\overline{\widehat{P}} = \overline{\widehat{Q}} \circ (\otimes^m T) : \otimes_\epsilon^m E \longrightarrow G$$

is Pietsch integral. By [V, Proposition 2.6], $\widehat{P}$ is Pietsch integral and, by [A2, Proposition 2], so is P. ∎

It is proved in [M, Proposition 3.1] that a polynomial $P \in \mathcal{P}(^m E, F)$ is r-dominated if and only if there are a constant $C \geq 0$ and a regular Borel probability measure μ on B_{E^*} (endowed with the weak-star topology) such that

$$\|P(x)\| \leq C \left[\int_{B_{E^*}} |\varphi(x)|^r \, d\mu(\varphi) \right]^{\frac{m}{r}} \qquad (x \in E). \tag{1}$$

The next result is stated in [P, Theorem 14] for the multilinear, scalar-valued case, and in [S, Proposition 3.6] for the vector-valued case. It will be needed in Proposition 6. Following the referee's suggestion, for the sake of completeness, we include the proof which is an easy modification of [G, 3.2.4].

Theorem 5. *A polynomial $P \in \mathcal{P}(^m E, F)$ is r-dominated if and only if there are a Banach space G, an absolutely r-summing operator $T \in \mathcal{L}(E,G)$ and a polynomial $Q \in \mathcal{P}(^m G, F)$ such that $P = Q \circ T$.*

Proof. Let $P \in \mathcal{P}(^m E, F)$ be r-dominated. Then there is a regular Borel probability measure μ on B_{E^*} such that the inequality (1) holds. Let $T_0 : E \to L_r(B_{E^*}, \mu)$ be given by $T_0(x)(\varphi) := \varphi(x)$ for all $x \in E$ and $\varphi \in B_{E^*}$. Clearly, T_0 is linear. Moreover,

$$\|T_0(x)\| = \left[\int_{B_{E^*}} |\varphi(x)|^r \, d\mu(\varphi)\right]^{\frac{1}{r}} \leq \|x\|.$$

Let G be the closure of $T_0(E)$ in $L_r(B_{E^*}, \mu)$. Let $T : E \to G$ be given by $T(x) := T_0(x)$. Then T is linear and, by [DJT, Theorem 2.12], absolutely r-summing. Define $Q_0 : T_0(E) \to F$ by $Q_0(T_0(x)) := P(x)$. Using the inequality (1), we have:

$$\|P(x)\| \leq C \left[\int_{B_{E^*}} |\varphi(x)|^r \, d\mu(\varphi)\right]^{\frac{m}{r}} = C\|T_0(x)\|^m,$$

so Q_0 is a continuous m-homogeneous polynomial. Let $Q : G \to F$ be its extension to G. Then, $P = Q \circ T$.

The converse is shown in [MT, Theorem 10]. ∎

The following result extends the linear case [DU, Theorem VI.3.12].

Proposition 6. *Let Ω be a compact Hausdorff space. Then, every 1-dominated polynomial $P \in \mathcal{P}(^m C(\Omega), E)$ is Pietsch integral.*

Proof. Let $P \in \mathcal{P}(^m C(\Omega), E)$ be 1-dominated. By Theorem 5, there are a Banach space G, an absolutely summing operator $T \in \mathcal{L}(C(\Omega), G)$ and a polynomial $Q \in \mathcal{P}(^m G, E)$ such that $P = Q \circ T$. By [DU, Theorem VI.3.12], T is Pietsch integral. By Proposition 4, P is a Pietsch integral polynomial. ∎

For a Banach space E, we denote by $\delta_m : E \to \otimes_{\pi,s}^m E$ the polynomial given by

$$\delta_m(x) = x \otimes \overset{(m)}{\cdots} \otimes x.$$

With each absolutely summing operator, the following lemma associates an m-homogeneous 1-dominated polynomial.

Lemma 7. *Let $T \in \mathcal{L}(E, F)$ be absolutely summing. Then*

$$(\otimes^m T) \circ \delta_m : E \longrightarrow \otimes_{\pi,s}^m F$$

is an m-homogeneous 1-dominated polynomial.

Proof. Fix $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in E$. Then

$$\begin{aligned} \sum_{i=1}^n \|(\otimes^m T) \circ \delta_m(x_i)\|^{1/m} &= \sum_{i=1}^n \left\|T(x_i) \otimes \overset{(m)}{\cdots} \otimes T(x_i)\right\|^{1/m} \\ &= \sum_{i=1}^n \|T(x_i)\| \\ &\leq k \sup_{x^* \in B_{E^*}} \sum_{i=1}^n |x^*(x_i)| \end{aligned}$$

where we have used that T is absolutely summing. ∎

Proposition 8. *Given Banach spaces E and F, and $m \in \mathbb{N}$, suppose that we have $\mathcal{P}_{\mathrm{PI}}({}^mE, F) = \mathcal{P}_{\mathrm{N}}({}^mE, F)$. Then $\mathcal{L}_{\mathrm{PI}}(E, F) = \mathcal{L}_{\mathrm{N}}(E, F)$.*

Proof. Since $\mathcal{L}_{\mathrm{N}}(E, F) \subseteq \mathcal{L}_{\mathrm{PI}}(E, F)$ is always true, we only need to prove the other inclusion. Let $T \in \mathcal{L}_{\mathrm{PI}}(E, F)$. By [DU, page 168], there is a regular Borel measure μ on B_{E^*} and an operator $b : L_1(B_{E^*}) = L_1(B_{E^*}, \mu) \to F$ such that the following diagram commutes

$$\begin{tikzcd} E \arrow[r, "T"] \arrow[d, "i"'] & F \\ C(B_{E^*}) \arrow[r, "j"'] & L_1(B_{E^*}) \arrow[u, "b"'] \end{tikzcd}$$

where i and j are the natural inclusions.

Fix $\overline{x} \in E$ and consider $\overline{f} = ji(\overline{x})$. Choose $\overline{g} \in L_\infty(B_{E^*})$ with $\overline{g}(\overline{f}) = 1$. Observe that

$$[i^* j^*(\overline{g})](\overline{x}) = j^*(\overline{g})(i(\overline{x})) = \overline{g}(ji(\overline{x})) = \overline{g}(\overline{f}) = 1.$$

For every index $i = 1, \ldots, m-1$, we consider the operators (see [B, page 168]):

$$\pi_i : \otimes_{\pi,s}^{i+1} L_1(B_{E^*}) \longrightarrow \otimes_{\pi,s}^{i} L_1(B_{E^*})$$

given by

$$\pi_i(\otimes^{i+1} f) = \overline{g}(f)(\otimes^i f)$$

and

$$\pi_i' : \otimes_{\pi,s}^{i+1} E \longrightarrow \otimes_{\pi,s}^{i} E$$

given by

$$\pi_i'(\otimes^{i+1} x) = (i^* j^*(\overline{g}))(x)(\otimes^i x).$$

The polynomials

$$T \circ \pi_1' \circ \ldots \circ \pi_{m-1}' \circ \delta_m : E \longrightarrow F$$

and

$$b \circ \pi_1 \circ \ldots \circ \pi_{m-1} \circ (\otimes^m j) \circ \delta_m \circ i : E \longrightarrow F$$

coincide. Indeed, we have

$$\begin{aligned} T \circ \pi_1' \circ \ldots \circ \pi_{m-1}' \circ \delta_m(x) &= T \circ \pi_1' \circ \ldots \circ \pi_{m-1}'(\otimes^m x) \\ &= ((i^* j^*(\overline{g}))(x))^{m-1} T(x) \\ &= ((i^* j^*(\overline{g}))(x))^{m-1} bji(x) \end{aligned}$$

and, on the other hand,

$$\begin{aligned} b \circ \pi_1 \circ \ldots \circ \pi_{m-1} \circ (\otimes^m j) \circ \delta_m \circ i(x) &= b \circ \pi_1 \circ \ldots \circ \pi_{m-1}(\otimes^m ji(x)) \\ &= (\overline{g}(ji(x)))^{m-1} bji(x). \end{aligned}$$

Since j is absolutely summing, thanks to Lemma 7, the polynomial

$$(\otimes^m j) \circ \delta_m : C(B_{E^*}) \longrightarrow \otimes_{\pi,s}^m L_1(B_{E^*})$$

is 1-dominated. Now, by Proposition 6, $(\otimes^m j) \circ \delta_m$ is Pietsch integral and so are (by Proposition 2) the polynomials

$$b \circ \pi_1 \circ \ldots \circ \pi_{m-1} \circ (\otimes^m j) \circ \delta_m \circ i = T \circ \pi_1' \circ \ldots \circ \pi_{m-1}' \circ \delta_m.$$

By our hypothesis, the latter is also nuclear. As shown in [CDG, pages 120-121], this implies that T is nuclear and so we are done. ∎

Theorem 9. *For a Banach space E, the following assertions are equivalent:*
(a) *E is Asplund;*
(b) *for all $m \in \mathbb{N}$ and every Banach space F, we have $\mathcal{P}_{\mathrm{PI}}(^mE, F) = \mathcal{P}_{\mathrm{N}}(^mE, F)$;*
(c) *there is $m \in \mathbb{N}$ such that for every Banach space F, we have $\mathcal{P}_{\mathrm{PI}}(^mE, F) = \mathcal{P}_{\mathrm{N}}(^mE, F)$.*

Proof. (a) $\Rightarrow$ (b) is proved in [A2, Proposition 1].
(b) $\Rightarrow$ (c) is obvious.
(c) $\Rightarrow$ (a). It is enough to apply Proposition 8 and Theorem 1. ∎

It is shown in [CD] that (a) implies that the equality of (b) is an isometry.

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