Solutions to the mean curvature equation by fixed point methods

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Abstract

We give conditions on the boundary data, in order to obtain at least one solution for the problem (1) below, with H a smooth function. Our motivation is a better understanding of the Plateau's problem for the prescribed mean curvature equation.

1 Introduction

We consider the Dirichlet problem in the unit disc $B = \{(u,v) \in \mathbf{R^2}; u^2 + v^2 < 1\}$ for a vector function $X : \overline{B} \longrightarrow \mathbf{R^3}$ which satisfies the equation of prescribed mean curvature

$$\begin{cases} \Delta X = 2H\,(X)\,X_u \wedge X_v \text{ in } B \\ \\ X = g \text{ on } \partial B \end{cases} \tag{1}$$

where $X_u = \dfrac{\partial X}{\partial u}$, $X_v = \dfrac{\partial X}{\partial v}$, $\wedge$ denotes the exterior product in $\mathbf{R^3}$ and $H : \mathbf{R^3} \longrightarrow \mathbf{R}$ is a given continuous function. For $H \equiv H_0 \in \mathbf{R}$ and g non constant with $0 < |H_0|\,\|g\|_\infty < 1$ there are two variational solutions ([1], [3]). For H near H_0 in certain cases there exist also two solutions to the Dirichlet problem ([2], [6]). For H far from H_0 , under appropriated conditions on g and H it is possible to obtain more than two solutions ([4]).

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We will consider prescribed smooth H and giving conditions on the boundary data g, we will prove the existence of a solution to (1) by fixed point theorems.

The main result is the following theorem

Theorem 1. *Let be $H \in C^1(\mathbf{R^3})$ and $g \in W^{2,p}(B, \mathbf{R^3})$ small enough, there exists a solution $X \in W^{2,p}(B, \mathbf{R^3})$ with $p > 2$ of (1).*

Finally, we recall that (1) is motivated for a better understanding of the Plateau's problem of finding a surface with prescribed mean curvature H which is supported by a given curve in $\mathbf{R^3}$.

2 Solution by fixed point methods

The systems (2) and (3) below are equivalent to (1) with $X = X_0 + Y$

$$\begin{cases} \Delta X_0 = 0 & \text{in } B \\ X_0 = g & \text{on } \partial B \end{cases} \tag{2}$$

$$\begin{cases} \Delta Y = F(X_0, Y) & \text{in } B \\ Y = 0 & \text{on } \partial B \end{cases} \tag{3}$$

and F defined as

$$F(X_0, Y) = 2H(X_0 + Y)(X_{0u} \wedge Y_v + Y_u \wedge X_{0v} + Y_u \wedge Y_v + X_{0u} \wedge X_{0v}).$$

Searching a fixed point of (3), we find it thanks to a variant of the Schauder theorem. We will work in a specific convex subset of the Sobolev space $W^{1,p}(B, \mathbf{R^3})$. We can write (3) in the following way :

$$\begin{cases} L(X_0)Y = \sum_{i=1}^{2} F_i(X_0, Y) & \text{in } B \\ Y = 0 & \text{on } \partial B \end{cases} \tag{4}$$

where $L(X_0)$ is the linear elliptic operator

$$L(X_0)Y = \Delta Y - 2(A_1(X_0)Y_u + A_2(X_0)Y_v),$$

$$A_1(X_0)Y_u = H(X_0)Y_u \wedge X_{0v}$$

$$A_2(X_0)Y_v = H(X_0)X_{0u} \wedge Y_v,$$

and $F_i(X_0, Y)$ defined by

$$F_1(X_0, Y) = 2(H(X_0 + Y) - H(X_0))(X_{0u} \wedge Y_v + Y_u \wedge X_{0v})$$

$$F_2(X_0, Y) = 2H(X_0 + Y)(X_{0u} \wedge X_{0v} + Y_u \wedge Y_v).$$

To prove Theorem 1, we will use the following technical lemmas :

Lemma 2. *Let be $X_0 \in W^{2,p}(B, \mathbf{R^3})$ with $p > 2$, then there exists $C > 0$ such that for any $R \in (0,1)$, $\delta > 0$*

1. $\|F_i(X_0, Y_1)\|_{p/2} \le C\left(\|X_0\|_{1,p}^2 + \|Y_1\|_{1,p}^2\right).$
2. $\|F_i(X_0, Y_1) - F_i(X_0, Y_2)\|_{p/2} \le C\left(\|Y_1 - Y_2\|_{1,p}\right)$
   $Y_j \in W_0^{1,p}(B, \mathbf{R^3})$ $\|Y_j\|_{1,p} \le R \quad j = 1, 2.$

Proof. As $H \in C^1(\mathbf{R^3})$, $X_0 \in W^{1,\infty}(B, \mathbf{R^3}), Y_j \in L^\infty(B, \mathbf{R^3})$ and $Y_{ju}, Y_{jv} \in L^p(B, \mathbf{R^3})$ the proof follows. ■

Lemma 3. *Let be $X_0 \in W^{2,p}(B, \mathbf{R^3})$ with $p > 2$, then there exists $C > 0$ such that*

$$\|A_i(X_0)\|_\infty \le C.$$

Proof. As $H \in C^1(\mathbf{R^3})$ and $X_0 \in W^{1,\infty}(B, \mathbf{R^3})$, the proof follows immediately. ■

Proposition 4. *Let be $X_0 \in W^{2,p}(B, \mathbf{R^3})$ with $p > 2$ small enough, then there exist $R \in (0,1)$ such that the following problem*

$$\begin{cases} L(X_0)\,Y = \sum_{i=1}^{4} F_i\left(X_0, \overline{Y}\right) & \text{in } B \\ \\ Y = 0 & \text{on } \partial B \end{cases} \tag{5}$$

define a continuous map $\overline{Y} \to Y$ in the closed ball with radio R of $W_0^{1,p}(B, \mathbf{R^3})$. Furthermore its range is a compact set.

Proof. Let $\overline{Y} \in W_0^{1,p}(B, \mathbf{R^3})$ with $\left\|\overline{Y}\right\|_{1,p} \le R$. From (1), using theorem 9.15 and lemma 9.17 in [5], we have

$$\|Y\|_{2,p/2} \le C\left(\|X_0\|_{1,p}^2 + \left\|\overline{Y}\right\|_{1,p}^2\right),$$

and Sobolev immersions imply that

$$\|Y\|_{1,p} \le C\left(\|X_0\|_{1,p}^2 + \left\|\overline{Y}\right\|_{1,p}^2\right) \le C\left(\|X_0\|_{1,p}^2 + R^2\right).$$

Choice $\|X_0\|_{1,p}^2$ and R small enough, we obtain

$$\|Y\|_{1,p} \le R. \tag{6}$$

From lemma 2, it follows that the map is continuous in $\overline{Y}$ and from (6), using compact Sobolev immersions, we conclude that its range is a compact set. ■

In order to prove the theorem, it is necessary to show that a fixed point Y $\in W^{2,p}(B, \mathbf{R^3})$.

Proof of theorem 1 Let be $Y \in W_0^{1,p}(B, \mathbf{R^3})$ a fixed point of (5), then $Y \in W^{2,p}(B, \mathbf{R^3})$. It is easy to see that $Y \in W^{2,p/2}(B, \mathbf{R^3})$, and then we obtain that $F_i(X_0, Y) \in L^r(B, \mathbf{R^3})$, with p/2< $r \leq p$. In the same way, we conclude that $Y \in W^{2,r}(B, \mathbf{R^3})$ and the proof follows.

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