On the left linear Riemann problem in Clifford analysis

Swanhild Bernstein

Abstract

We consider a left-linear analogue to the classical Riemann problem:

$$\begin{aligned} D_a u &= 0 \text{ in } \mathbb{R}^n \backslash \Gamma \\ u^+ &= H(x)u^- + h(x) \text{ on } \Gamma \\ |u(x)| &= \mathcal{O}(|x|^{\frac{n}{2}-1}) \text{ as } |x| \to \infty. \end{aligned}$$

For this purpose, we state a Borel-Pompeiu formula for the disturbed Dirac operator $D_a = D + a$ with a paravector a and some functiontheoretical results. We reformulate the Riemann problem as an integral equation:

$$P_a u + HQ_a u = h \text{ on } \Gamma,$$

where $P_a = \frac{1}{2}(I + S_a)$ and $Q_a = I - P_a$. We demonstrate that the essential part of the singular integral operator S_a which is constructed by the aid of a fundamental solution of $D + a$ is just the singular integral operator S associated to D. In case S_a is simply S and $\Gamma = \mathbb{R}^{n-1}$, then under the assumptions

1. $H = \sum_\beta H_\beta e_\beta$ and all H_β are real-valued, measurable and essentially bounded;
2. $(1 + H(x))\overline{(1 + H(x))}$ and $H(x)\bar{H}(x)$ are real numbers for all $x \in \mathbb{R}^{n-1}$;
3. the scalar part H_0 of H fulfils $H_0(x) > \varepsilon > 0$ for all $x \in \mathbb{R}^{n-1}$,

the Riemann problem is uniquely solvable in $L_{2,\mathcal{C}}(\mathbb{R}^{n-1})$ and the successive approximation

$$u_n := 2(1 + H)^{-1}h - (1 + H)^{-1}(1 - H)Su_{n-1}, \quad n = 1, 2, \ldots,$$

Received by the editors July 1995.
Communicated by R. Delanghe.
1991 *Mathematics Subject Classification* : 35F15; 47B.
Key words and phrases : Riemann problem, Singular Cauchy-type operators.

with arbitrary $u_0 \in L_{2,\mathcal{C}}(\mathbb{R}^{n-1})$ converges to the unique solution of

$$Pu + HQu = \frac{1}{2}(1 + H)u + \frac{1}{2}(1 - H)Su = h.$$

Further, we demonstrate that the adjoint operator $S_a^* = nS_{-a^\star}n$ and describe dense subsets of im P_a and im Q_a using orthogonal decompositions. We apply our results to Maxwell's equations.

1 Introduction

The following boundary value problem was first formulated by Riemann in his inaugural dissertation. Since a first attempt towards a solution was made by Hilbert through the use of integral equations, what we will denote as Riemann problem is in the literature sometimes also called Hilbert or Riemann-Hilbert problem.

Riemann problem

Let $\mathbb{D}_-$ be a bounded and simply connected domain in the complex plane and denote by $\mathbb{D}_+ := \mathbb{C}\backslash\overline{\mathbb{D}}_-$ its unbounded open complement. Then the Riemann problem consists in finding a function f which is holomorphic in $\mathbb{D}_-$ and $\mathbb{D}_+$, which can be continuously extended from $\mathbb{D}_+$ into $\overline{\mathbb{D}}_+$ and from $\mathbb{D}_-$ into $\overline{\mathbb{D}}_-$ satisfying the boundary condition

$$f_- = Hf_+ + h \text{ on } \Gamma \text{ and } f(z) \to 0, \quad z \to \infty,$$

uniformly for all directions. Here, H and h are given functions on Γ.

Clifford generalization

Let G be a bounded and simply connected domain in $\mathbb{R}^n$ which is bounded by a Liapunov surface $\partial G = \Gamma$. Then we want to investigate the following Clifford analogue to the classical Riemann problem :

$$\begin{aligned} D_a u &= 0 \text{ in } \mathbb{R}^n\backslash\Gamma \\ u^+ &= H(x)u^- + h(x) \text{ on } \Gamma \\ |u(x)| &= \mathcal{O}(|x|^{\frac{n}{2}-1}) \text{ as } |x| \to \infty, \end{aligned}$$

where D_a denotes the disturbed Dirac operator $D + a$, a being a paravector. Using the Plemelj-Sokhotzki formulae this problem can be transformed into a singular integral equation on the boundary Γ :

$$P_a u + H(x)Q_a u = h(x).$$

The linear Riemann problem in the complex plane is well known, especially the situation where $\mathbb{D}_-$ is the unit disk. The main tools here are the existence of a simple orthonormal system on the unit circle, namely e^{int}, $n \in \mathbb{N}$, and a multiplier theorem. An analogous orthonormal system can be constructed for the unit sphere (see [4] or [6]). Unfortunately, it has a very complex structure. A similar multiplier theorem does not exist. Hence, we have to look for other methods.

Fredholm properties of the singular integral equation which is equivalent to the Riemann problem are investigated by *Shapiro* and *Vasilevski* in [17]. But in this paper there are no results about computing a solution and about the uniqueness of the solution.

Successive approximation for the singular integral equation in spaces of Hölder continuos functions was considered by *Xu* in [21]. However in these spaces it is difficult to compute the norm of the singular integral operator S. Consequently all results in [21] involve an unknown constant caused by the norm of the operator S. On the other hand successive approximation is used straightforwardly such that solvability is stated only for functions H with "small" norm.

The best situation for the Riemann problem in Clifford analysis is when G is the upper or lower half space $\mathbb{R}^n$ and the singular integral operator S in the Hilbert module $L_{2,\mathcal{C}}$ is considered. Then the operator S is unitary and the norm is obviously 1. In this case we extract an uncomplicated sufficient condition for the unique solvability of the Riemann problem. Namely, if $H = \sum_\beta H_\beta$ and all H_β are real-valued, measurable and essentially bounded, $(1 + H(x))\overline{(1 + H(x))}$ is a real number for all x and the scalar part of H_0 of H fulfils $H_0(x) > \varepsilon > 0$ for all $x \in \mathbb{R}^{n-1}$ then the Riemann problem is uniquely solvable in $L_{2,\mathcal{C}}$ by successive approximation.

In the first sections of this paper we state some function theoretical results for the Dirac operator D_a. The most important fact is the Borel-Pompeiu or Cauchy-Green formula. The situation where a is a complex number is fully discussed by *Xu* in [20]. In the quaternionic algebra the formula was proved by *Kravcenko* in [9]. We take another proof based on the method used by *Gürlebeck* and *Sprößig* for the operator D in [8]. From the Borel-Pompeiu formula we obtain the Cauchy formula and Lusin's theorem as was done in the case of D by *Brackx*, *Delanghe* and *Sommen* in [4].

In the general situation with the operator S_a related to the Dirac operator D_a we will see that the main part of the operator is simply S. Unfortunatly, we are not able to compute the norm of the singular integral operator S_a in $L_{2,\mathcal{C}}$. Nevertheless, taking into account orthogonality in the Hilbert module $L_{2,\mathcal{C}}$ we describe dense subsets of the set of functions defined on the boundary which can be continuously extended to functions of $\ker D_a$ in the domain G and its complement $\mathbb{R}^n \backslash \overline{G}$ respectivly. Here, we generalize a result of [8] and simplify the proof given there.

Finally, we discuss Maxwell's equations in the Clifford algebra $\mathcal{C}_{0,3}$ as an application of our considerations.

2 Preliminaries

Let $\mathbf{R}_{0,n}$ be the real Clifford algebra with generating vectors $e_i,\ i,\dots,n$, where, $e_i^2 = -1$, and $e_ie_j + e_je_i = 0$ if $i \neq j$ and $i,j = 1,2\dots,n$. Besides, let e_0 be the unit element. Further, let $\mathcal{C}_{0,n} = \mathbf{R}_{0,n} \otimes \mathbb{C}$ be the associated complex Clifford algebra. Then an arbitrary element $b \in \mathcal{C}_{0,n}$ is given by $b = \sum_\beta b_\beta e_\beta$, where $b_\beta \in \mathbb{C}$ and $e_\beta = e_{\beta_1}\cdot e_{\beta_2}\cdot\ldots\cdot e_{\beta_h},\ \beta_1,\beta_2,\ldots\beta_h \in \{1,...,n\}$ and $\beta_1 < \beta_2 < \ldots < \beta_h$. A conjugation is defined by $\overline{b} = \sum_\beta b_\beta \overline{e_\beta},\ \overline{e_\beta} = \overline{e_{\beta_h}}\cdot\ldots\cdot\overline{e_{\beta_2}}\ \cdot\overline{e_{\beta_1}}$ and $\overline{e_0} = e_0,\ \overline{e_j} = -e_j,\ j = 1,\dots,n$. By $[b]_0 = b_0e_0$ we denote the scalar part of b, whereas $Im\ b = \sum_{\beta,\beta\neq 0} b_\beta e_\beta$ denotes the imaginary or multivector part.

The following facts of Clifford algebras are contained in [7]. In the Clifford-algebra $\mathcal{C}_{0,n}$ we introduce a general substitute for the determinant. This is the norm function

$$\triangle : \mathcal{C}_{0,n} \to \mathcal{C}_{0,n}, \quad \triangle(x) = \overline{x}x.$$

Since $\overline{\lambda x} = \lambda\overline{x}$ for all $\lambda \in \mathbb{C}$ and $x \in \mathcal{C}_{0,n}$, clearly $\triangle(\lambda x) = \lambda^2 \triangle(x)$; but, in general, $\triangle$ is not a quadratic form on $\mathcal{C}_{0,n}$ since $\triangle$ is not $\mathbb{C}$-valued on $\mathcal{C}_{0,n}$ when $n > 2$. For instance, take $n \geq 3$ and $x = 1 + e_1e_2e_3$; then $x = \overline{x}$ and

$$\triangle(x) = x^2 = 1 + 2e_1e_2e_3 + (e_1e_2e_3)^2 = 2 + 2e_1e_2e_3 \notin \mathbb{C}.$$

Nevertheless, on the subset

$$\mathcal{N} = \{x \in \mathcal{C}_{0,n} : \triangle(x) \in \mathbb{C}\backslash\{0\}\}$$

on which $\triangle$ is $\mathbb{C}$-valued the basic properties of the norm function mirror those of the determinant.

Theorem 1 *(cf. [7]) The set $\mathcal{N}$ is a multiplicative subgroup in $\mathcal{C}_{0,n}$ which is closed under scalar multiplication as well as under conjugation. Furthermore*

$$\triangle(xy) = \triangle(x) \triangle(y), \quad \triangle(x) = \triangle(\overline{x}), \quad x,y \in \mathcal{N},$$

and for x in $\mathcal{N}$

$$x^{-1} = \frac{1}{\triangle(x)}\overline{x}, \qquad \triangle(x^{-1}) = \frac{1}{\triangle(x)}.$$

Additionally we introduce a complex-conjugation by on $\mathcal{C}_{0,n}$ as follows :

$$\text{if} \qquad b_\alpha = b_\alpha^{(1)} + ib_\alpha^{(2)}$$

$$\text{then we put} \quad b_\alpha^\star = b_\alpha^{(1)} - ib_\alpha^{(2)} \quad \text{and} \quad b^\star = \left(\sum_\alpha b_\alpha e_\alpha\right)^\star = \sum_\alpha {b_\alpha}^\star e_\alpha, \quad b \in \mathcal{C}_{0,n},$$

where i denotes the imaginary unit.

The sesqui-linear inner product on $\mathcal{C}_{0,n}$ is given by

$$(u,v)=\left(\sum_\alpha u_\alpha e_\alpha,\sum_\beta v_\beta e_\beta\right)=\sum_\alpha u_\alpha v_\alpha^\star$$

and the Hilbert space norm by

$$|u|=\left|\sum_\alpha u_\alpha e_\alpha\right|=\left(\sum_\alpha |u_\alpha|^2\right)^{\frac{1}{2}}$$

They coincide with the usual Euclidean ones on $\mathcal{C}_{0,n}$ as a space of complex dimension 2^n. This Hilbert space norm will be said to be the Clifford norm on $\mathcal{C}_{0,n}$. But because this norm is not submultiplicative, we introduce the Clifford operator norm

$$|b|_{Op}=\sup\{|bu|:u\in\mathcal{C}_{0,n},\ |u|\le 1\}$$

that has the submultiplicative property $|ab|_{Op}\le|a|_{Op}|b|_{Op}$ $\forall a,b\in\mathcal{C}_{0,n}$. Now, $\mathcal{C}_{0,n}$ becomes equipped with an involution: the Clifford conjugation. But this involution is complex linear, not conjugate-linear. So for B* algebra purposes, we shall take as involution on $\mathcal{C}_{0,n}$ the conjugate-linear extension $b\to\widetilde{b}$ of conjugation on real Clifford algebras. Thus

$$\widetilde{b}=\widetilde{\left(\sum_\alpha b_\alpha e_\alpha\right)}=\sum_\alpha {b_\alpha}^\star(-1)^{\frac{1}{2}|\alpha|(|\alpha|+1)}e_\alpha,$$

whereas

$$\overline{b}=\overline{\sum_\alpha b_\alpha e_\alpha}=\sum_\alpha b_\alpha(-1)^{\frac{1}{2}|\alpha|(|\alpha|+1)}e_\alpha.$$

Theorem 2 *(cf. [7]) Under the involution $b\to\widetilde{b}$ and the Clifford operator norm, $\mathcal{C}_{0,n}$ is a complex C^*-algebra.*

Lemma 1 *(cf. [7]) We have the following usefull relations :*

(i) $(au,v)=(u,\widetilde{a}v)$

(ii) $|aub|\le|a|_{Op}|u||b|_{Op}$

for all $a,b,$ and u in $\mathcal{C}_{0,n}$.
In particular if a is an element of the real Clifford algebra $\mathbf{R}_{0,n}$,

(iii) $|a|^2=\triangle(a)=|a|^2_{Op}$

whenever the 'norm' $\triangle(a)=\overline{a}a$ is real-valued.

Furthermore, we identify an $x\in\mathbb{R}^n$ with $x=\sum\limits_{i=1}^n x_ie_i\in\mathcal{C}_{0,n}$.

3 Function spaces

A function $u = \sum_\beta u_\beta e_\beta$ belongs to the function space $F_\mathcal{C}$ iff all real-valued functions u_β belong to the function space F of real-valued functions.
We denote by $C^{0,\alpha}_\mathcal{C}(\Gamma),\ 0 < \alpha < 1$, the space of Hölder-continuous and by $C_\mathcal{C}(\Gamma)$ the space of continuous functions. Additionally $L_{\infty,\mathcal{C}}(\Gamma)$ is the space of all measurable essentially bounded functions on Γ.
We also consider the right-Hilbert-module $L_{2,\mathcal{C}}(\Gamma)$, with the inner product

$$(u, v) = \int_\Gamma \widetilde{u} v d\Gamma$$

which leads to the norm

$$||u||^2_{L_{2,\mathcal{C}}} = [(u, u)]_0.$$

For more details see [4].

4 Dirac-type operators and Cauchy-type integrals

We introduce the following operators :
the Dirac operator

$$D = \sum_{i=1}^n e_i \frac{\partial}{\partial x_i}$$

and the Dirac-type operator

$$D_a = D + a,$$

where a is a paravector, this means $a = \sum_{j=0}^n a_j e_j,\ a_j \in \mathbb{C}$.
A fundamental solution of D_a is given by (cf. [2])

$$-E_a(x) = -e^{[ax]_0}\{(D - a_0)K_{ia_0}(x)\},$$

where

$$K_{ia_0}(x) = K_{ia_0}(|x|) = \frac{-1}{(2\pi)^{\frac{n}{2}}} \left(\frac{ia_0}{|x|}\right)^{\frac{n}{2}-1} K_{\frac{n}{2}-1}(ia_0|x|)$$

and $K_{\frac{n}{2}-1}$ denotes a Bessel-function of third order, the so-called MacDonalds-function. Note that $K_{ia_0}(|x|)$ is a fundamental solution of $\triangle + a_0^2$ (cf. [14]). With this fundamental solution we define

$$(T_a u)(x) = \int_G -E_a(x - y)u(y)dG,$$

$$(F_a u)(x) = \int_\Gamma E_a(x - y)n(y)u(y)d\Gamma,\ x \notin \Gamma,$$

where $n(y)$ denotes the outward unit normal on Γ at the point y.
Furthermore, we put

$$(S_a u)(x) = 2\int_\Gamma E_a(x-y)n(y)u(y)d\Gamma,\ x\in\Gamma,$$

and introduce the algebraic projections

$$P_a = \tfrac{1}{2}(I+S_a) \text{ and } Q_a = \tfrac{1}{2}(I-S_a).$$

If $a=0$ we write $E(x)$ instead of $E_0(x)$ an also S instead of S_a, etc. .... In particular we have

$$E(x) = \frac{1}{A_n}\frac{x}{|x|^n},\quad A_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}.$$

5 Function theoretic results

The first principal application of our considerations is the following:

Lemma 2 *Let $u\in C_{\mathcal{C}}(\bar{G})$, then*

$$(D_aT_au)(x) = \begin{cases} u(x),\ x\in G \\ 0,\quad x\in\mathbb{R}^n\backslash\overline{G}. \end{cases}$$

Proof: We remark that $-E_a(x)$ is a fundamental solution of D_a. ∎

Theorem 3 (Cauchy-Green or Borel-Pompeiu formula) *For $u\in C^1_{\mathcal{C}}(G)\cap C_{\mathcal{C}}(\bar{G})$ we have*

$$(F_au)(x) + (T_aD_au)(x) = \begin{cases} u(x)\ in\ \ G \\ 0\ in\ \mathbb{R}^n\backslash\bar{G} \end{cases}.$$

In particular we obtain a Cauchy-type formula

Lemma 3 (Cauchy formula) *For $u\in C^1_{\mathcal{C}}(G)\cap C_{\mathcal{C}}(\bar{G})$ and $D_au(x)=0$ in G we have*

$$(F_au)(x) = u(x)\ in\ \ G$$

and $(F_au)(x)=0$ in G iff $u(x)=0$ in G.

Thus functions of the kernel of D_a are uniquely determined by their boundary values. An effect of the Borel-Pompeiu formula is the following theorem.

Theorem 4 *Let $u\in C_{\mathcal{C}}(\mathbb{R}^n)$ satisfy $(\triangle + 2\sum\limits_{j=1}^{n} a_j\frac{\partial}{\partial x_j} + \overline{a}a)u = 0$, where a is a paravector $a=\sum\limits_{k=0}^{n} a_ke_k$. Then u can be represented as*

$$u = \varphi + \psi$$

where $\psi\in\ker(D+a)$ in G and $\varphi\in\ker(D+a)$ in $\mathbb{R}^n\backslash\overline{G}$.

Proof: Applying the Borel-Pompeiu-formula twice we get

$$T_a T_{-\overline{a}}(\triangle + 2\sum_{j=1}^{n} a_j \frac{\partial}{\partial x_j} + \overline{a}a)u = -T_a T_{-\overline{a}}(D - \overline{a})(D + a)u =$$
$$-T_a(-F_{-\overline{a}}(D + a)u + (D + a)u) = T_a F_{-\overline{a}}(D + a)u + F_a u - u = 0.$$

Hence

$$u = F_a u + T_a F_{-\overline{a}}(D + a)u,$$

where $F_a u \in \ker(D + a)$ in G and using the Borel-Pompeiu's formula again, $T_a F_{-\overline{a}}(D + a)u = -F_a T_a(F_{-\overline{a}}(D + a)u) \in \ker(D + a)$ in $\mathbb{R}^n \backslash \overline{G}$. ■

Theorem 5 (Plemelj-Sokhotzki formulae) *For $u \in C_{\mathcal{C}}^{0,\alpha}(G)$, $0 < \alpha < 1$, we have*

$$\lim_{G \ni x \to x_0 \in \Gamma} (F_a u)(x) = \frac{1}{2}\{u(x_0) + (S_a u)(x_0)\} = (P_a u)(x_0)$$
$$\lim_{\mathbb{R}^n \backslash \bar{G} \ni x \to x_0 \in \Gamma} (F_a u)(x) = -\frac{1}{2}\{u(x_0) - (S_a u)(x_0)\} = -(Q_a u)(x_0).$$

An important consequence of the Plemelj-Sokhotzki formulae is the identity $S_a^2 = I$. These results can be extended to functions of $L_{2,\mathcal{C}}$.

Theorem 6 (Lusin's theorem) *Let $u \in C_{\mathcal{C}}^1(G) \cap C_{\mathcal{C}}(\overline{G})$ and $D_a u = 0$ in $G \subset \mathbb{R}^n$. Further, let $\gamma \subset \Gamma$ be a (n-1)-dimensional submanifold and $u(x) = 0$ on γ. Then $u(x) = 0$ in $\overline{G}$.*

Lemma 4 *Let $u \in C_{\mathcal{C}}(\overline{G})$ and $D_a u = 0$ in G and $u = 0$ on a (n-1)-dimensional submanifold $\gamma \in \overline{G}$. Then u is identically zero in $\overline{G}$.*

The proofs may be found in [4, 6, 8, 11, 20].

6 Singular Cauchy-type operators

In this section we want to state some important properties of the operator S_a. First we demonstrate that the essential part of the operator S_a is just the operator S.

Theorem 7 *The operator $S : L_{2,\mathcal{C}}(\Gamma) \to L_{2,\mathcal{C}}(\Gamma)$ is continuous.*

The proof can be found e.g. in [8], [10], [11].

Theorem 8 *The operator $S_a - S$ is compact in $L_{2,\mathcal{C}}(\Gamma)$.*

Proof: We have

$$(S_a u)(x) = 2 \int_\Gamma E_a(x-y)n(y)u(y)d\Gamma,\ x \in \Gamma,$$

$$(Su)(x) = 2 \int_\Gamma E(x-y)n(y)u(y)d\Gamma,\ x \in \Gamma,$$

where

$$E(x-y) = \frac{1}{A_n}\frac{x-y}{|x-y|^n}$$

and

$$E_a(x) = e^{[ax]_0}\{(D-a_o)K_{ia_0}(x)\}$$
$$= e^{[ax]_0}\left\{\frac{1}{(2\pi)^{\frac{n}{2}}}\sum_{j=1}^{n}\frac{x_j e_j}{|x|^n}(ia_0|x|)^{\frac{n}{2}}K_{\frac{n}{2}}(ia_0|x|) + \frac{ia_0}{(2\pi)^{\frac{n}{2}}}\frac{1}{|x|^{n-2}}(ia_0|x|)^{\frac{n}{2}-1}K_{\frac{n}{2}-1}(ia_0|x|)\right\}.$$

Using the properties of modified Bessel functions we get that

$$e^{[ax]_0}\frac{1}{(2\pi)^{\frac{n}{2}}}(ia_0|x|)^{\frac{n}{2}}K_{\frac{n}{2}}(ia_0|x|) = \frac{1}{A_n} + \mathcal{O}(|x|^\alpha),\ \alpha > 0,\ \text{as } x \to 0.$$

Thus

$$E_a(x) - E(x) = \left\{e^{[ax]_0}\frac{1}{(2\pi)^{\frac{n}{2}}}(ia_0|x|)^{\frac{n}{2}}K_{\frac{n}{2}}(ia_0|x|) - \frac{1}{A_n}\right\}\frac{x_j e_j}{|x|^n} +$$
$$+ e^{[ax]_0}\frac{ia_0}{(2\pi)^{\frac{n}{2}}}\frac{1}{|x|^{n-2}}(ia_0|x|)^{\frac{n}{2}-1}K_{\frac{n}{2}-1}(ia_0|x|)$$

As the kernel $E_a(x) - E(x)$ is weakly singular, the corresponding integral operator $S_a - S$ is compact. ■

Theorem 9 *The operator* $S_a\ :\ L_{2,\mathcal{C}}(\Gamma) \to L_{2,\mathcal{C}}(\Gamma)$ *is continuous.*

Proof: We write $S_a = S + (S_a - S)$. As S is continuous and $S_a - S$ is compact, S_a is also continuous. ■

Remark 1 *Theorems 7, 8 and 9 are also valid in the spaces* $L_{p,\mathcal{C}}(\Gamma),\ 1 < p < \infty$.

7 Successive Approximation

In this section we demonstrate that under weak conditions the successive approximation for the left-linear Riemann problem converges. This means that the problem is uniquely solvable for all right hand sides. We rewrite our problem into the form

$$Au = P_a u + HQ_a u = \tfrac{1}{2}(1+H)u + \tfrac{1}{2}(1-H)S_a u = h,$$

where $H \in L_{\infty,\mathcal{C}}(\Gamma)$. If $(1+H)$ is invertible on Γ then we can investigate the problem

$$u + (1+H)^{-1}(1-H)S_a u = 2(1+H)^{-1}h.$$

An immediate outcome appears in the situation where $H(x) = \sum_\beta H_\beta(x)$, all H_β being real-valued, G is the lower half space $\mathbb{R}^n_-$ and S_a is simply S.

Theorem 10 *Let G be the lower half space $\mathbb{R}^n_-$ and assume that*

(i) $H(x) = \sum_\beta H_\beta(x) e_\beta$, *and all* H_β *are real-valued;*

(ii) $(1 + H(x))\overline{(1 + H(x))} \in \mathbb{R}$ *and* $H(x)\overline{H}(x) \in \mathbb{R}$ *for all* $x \in \mathbb{R}^{n-1}$

and

(iii) there exists an $\varepsilon > 0$ *with* $0 < \varepsilon < 1$ *such that* $H_0(x) > \varepsilon$ *for all* $x \in \mathbb{R}^{n-1}$.

Then the Riemann problem

$$Au = Pu + HQu = \tfrac{1}{2}(1 + H)u + \tfrac{1}{2}(1 - H)Su = h$$

is uniquely solvable in $L_{2,\mathcal{C}}(\mathbb{R}^{n-1})$ *and the successive approximation*

$$u_n := 2(1 + H)^{-1}h - (1 + H)^{-1}(1 - H)Su_{n-1}, \quad n = 1, 2, \ldots$$

with arbitrary $u_0 \in L_{2,\mathcal{C}}(\mathbb{R}^{n-1})$ *converges to the unique solution* u *of*

$$Au = Pu + HQu = \tfrac{1}{2}(1 + H)u + \tfrac{1}{2}(1 - H)Su = h.$$

Proof: We have to show that

$$\sup\operatorname{ess} |(1 + H)^{-1}(1 - H)|_{Op} < ||S||^{-1}.$$

In Section 9 Lemma 11 we will find out that the Cauchy-type singular integral operator S is unitary in $L_{2,\mathcal{C}}(\mathbb{R}^{n-1})$ and thus $||S||_{L_{2,\mathcal{C}}} = 1$. Accordingly, we have to prove

$$\sup\operatorname{ess} |(1 + H)^{-1}(1 - H)|_{Op} < 1.$$

As stated in Theorem 1 and Lemma 1 we can simplify $|(1 + H)^{-1}(1 - H)|_{Op}$ into $|(1 + H)^{-1}(1 - H)|$ if $(1 + H)^{-1}(1 - H)\overline{(1 + H)^{-1}(1 - H)} \in \mathbb{R}$.
By assumption (ii) $(1 + H)\overline{(1 + H)} \in \mathbb{R}$ whence $(1 + H)\overline{(1 + H)} = 1 + 2H_0 + |H|^2$.
As moreover $(1 + H)\overline{(1 + H)} + (1 - H)\overline{(1 - H)} = 2 + 2H\overline{H}$ we obtain by (ii) that $(1 - H)\overline{(1 - H)} \in \mathbb{R}$.

If $H_0(x) > \varepsilon > 0$ then $\triangle(1 + H) > 1 + 2\varepsilon > 0$. Therefore, we get

$$(1 + H(x))^{-1} = \frac{(1 + \overline{H(x)})}{|1 + H(x)|^2}.$$

Moreover $(1 - H)(1 - \overline{H}) = 1 - H - \overline{H} + H\overline{H} = 1 - 2H_0 + |H|^2$ and consequently

$$(1 + H)^{-1}(1 - H)\overline{(1 + H)^{-1}(1 - H)} = \frac{(1 + \overline{H})(1 - H)(1 - \overline{H})(1 + H)}{|1 + H|^4}$$

is real valued on $\mathbb{R}^{n-1}$. Finally, as by (ii), $Im(H + \overline{H}) = 0$ and $Im(H\overline{H}) = 0$

$$
\begin{aligned}
|(1+H)^{-1}(1-H)|^2 &= \left|\frac{(1+\overline{H})(1-H)}{(1+H)(1+\overline{H})}\right|^2 = \left|\frac{1+\overline{H}-H-H\overline{H}}{1+\overline{H}+H+H\overline{H}}\right|^2 \\
&= \left|\frac{1-2Im\ H-|H|^2}{1+2H_0+|H|^2}\right|^2 = \frac{(1-|H|^2-2Im\ H)(1-|H|^2+2Im\ H)}{(1+2H_0+|H|^2)^2} \\
&= \frac{(1-|H|^2)^2+4Im\ H Im\ \overline{H}}{(1+|H|^2+2H_0)^2}
\end{aligned}
$$

If there exists an $\varepsilon > 0$ such that the last expression is less than $1-\varepsilon$ on $\mathbb{R}^{n-1}$ then the condition for the convergence of the successive approximation is fulfilled. Assume

$$
\begin{aligned}
&\frac{(1-|H|^2)^2+4Im\ H Im\ \overline{H}}{(1+|H|^2+2H_0)^2} < 1-\varepsilon \iff \\
&1-2|H|^2+|H|^4+4|Im\ H|^2 < (1-\varepsilon)\{(1+|H|^2)^2+4H_0(|H|^2+1)+4(H_0)^2\}
\end{aligned}
$$

We set

$$\tilde{\varepsilon} := \varepsilon(1+|H|^2+2H_0)^2 \leq \varepsilon \sup\text{ess}(1+|H|^2+2H_0)^2;$$

then $\tilde{\varepsilon}$ is an arbitrary positive real number, because $\varepsilon > 0$ was arbitrary chosen. Hence

$$
\begin{aligned}
&1-2|H|^2+|H|^4+4|Im\ H|^2 < 1+2|H|^2+|H|^4+4H_0|H|^2+4H_0+4(H_0)^2-\tilde{\varepsilon} \\
\iff& 4|Im\ H|^2 < 4|H|^2+4H_0|H|^2+4H_0+4(H_0)^2-\tilde{\varepsilon} \\
\iff& 4|Im\ H|^2 < 4(H_0)^2+4|Im\ H|^2+4H_0|H|^2+4H_0+4(H_0)^2-\tilde{\varepsilon} \\
\iff& 0 < 4H_0(2H_0+|H|^2+1)-\tilde{\varepsilon} \\
\iff& \tilde{\varepsilon} < 4H_0|1+H|^2
\end{aligned}
$$

This condition is fulfilled if (iii) holds. ■

Remark 2 *The Theorem above tells us that the index of the Riemann problem is zero under the assumptions made. Especially we need* $\inf H_0(x) > 0$. *In [3] we prove that the index of the Riemann problem is zero not only in this situation.*

8 Examples

Example 1 Let G be the upper half plane $\mathbb{R}^2_+$. Then $\partial\mathbb{R}^2_- = \mathbb{R}^1$. Then we can use the Clifford algebra $\mathbf{R}_{0,2}$ with the generating vectors e_1, e_2 to create the singular Cauchy-type integral operator

$$Su(x) = -\frac{1}{\pi}\int_{\mathbb{R}^1} \frac{(x-y)}{|x-y|^2} e_1 e_2 u(x)dy,$$

with

$$u(x) = u_0(x)e_0 + u_1(x)e_1 + u_2(x)e_2 + u_{12}(x)e_1e_2.$$

We consider the operator $Au = Pu + HQu$ of the Riemann problem where H has the same structure as u and we suppose each $H_\beta(x)$ to be real-valued. This one-dimensional problem can be compared with the classical situation by setting

$$\begin{aligned} u(x) &= u_0(x)e_0 + u_{12}e_1e_2 = v(x) + iw(x), \\ H(x) &= H_0(x)e_0 + H_{12}e_1e_2 = F(x) + iG(x). \end{aligned}$$

Thus, the simple condition

$$H_0(x) > \tilde{\varepsilon} > 0 \quad \forall x$$

is sufficient for the convergence of the successive approximation.

Example 2 Let G be the upper half space $\mathbb{R}^3_+$. Then $\partial\mathbb{R}^3_- = \mathbb{R}^2$. Then we can use the Clifford algebra $\mathbf{R}_{0,2}$ (Quaternions) with the generating vectors e_1, e_2 to create the singular Cauchy-type integral operator

$$Su(x) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{(x_1-y_1)e_1 + (x_2-y_2)e_2}{|x-y|^3} e_{12} u(x)dy,$$

with

$$u(x) = u_0(x)e_0 + u_1(x)e_1 + u_2(x)e_2 + u_{12}(x)e_{12}.$$

We consider the operator $Au = Pu + HQu$ of the Riemann-problem where H has the same structure as u and we suppose each $H_\beta(x)$ to be real-valued, $H(x)\overline{H(x)}$ and $(1+H(x))\overline{(1+H(x))}$ are real numbers for all $x \in \mathbb{R}^{n-1}$. We get as a sufficient condition for the convergence of the successive approximation $H_0(x) > \varepsilon > 0 \;\; \forall x$.

Example 3 Let G be again the upper half space $\mathbb{R}^3_+$. Then $\partial\mathbb{R}^3_- = \mathbb{R}^2$. Then we use the Clifford algebra $\mathbf{R}_{0,3}$ with the generating vectors e_1, e_2, e_3 and the singular Cauchy-type integral operator

$$Su(x) = -\frac{1}{2\pi}\int\limits_{\mathbb{R}^2} \frac{(x_1-y_1)e_1+(x_2-y_2)e_2}{|x-y|^3} e_3 u(x)dy,$$

with

$u(x) =$
$u_0(x)e_0 + u_1(x)e_1 + u_2(x)e_2 + u_3(x)e_3 + u_{12}(x)e_{12} + u_{13}(x)e_{13} + u_{23}(x)e_{23} + u_{123}e_{123}.$

We consider the operator $Au = Pu + HQu$ of the Riemann-problem where H has the same structure as u and we suppose each $H_\beta(x)$ to be real-valued. To obtain a simple criterion for the successive approximation we suppose $H(x)\overline{H(x)}$ and $(1 + H(x))\overline{(1+H(x))}$ to be real numbers for all $x \in \mathbb{R}^{n-1}$. Then we get that the condition $H_0(x) > \varepsilon > 0 \;\; \forall x$ is sufficient for the convergence of the successive approximation.

9 Projections and orthogonal decompositions

It is easily seen that P_a and Q_a are algebraic projections. But they are not orthogonal in the sense of the inner product $(.,.)$ of $L_{2,\mathcal{C}}(\Gamma)$. Because P_a and Q_a are idempotent we have orthogonal decompositions

$$L_{2,\mathcal{C}}(\Gamma) = \ker P_a^* \oplus \text{ im } P_a,$$
$$L_{2,\mathcal{C}}(\Gamma) = \ker Q_a^* \oplus \text{ im } Q_a,$$

where the star denotes the adjoint operator. Thus we are interested in the adjoint operators. The following lemma helps us to construct the operator S_a^*.

Lemma 5 *Let z be a complex variable. Then we have $(K_\nu(iz))^\star = K_\nu((iz)^\star)$ for all $\nu \in \mathbb{R}$.*

Proof: We mention that the star $\star$ denotes complex conjugation. According to [1] we have for a complex variable z and a real ν

$K_\nu(iz) = \frac{1}{2}i\pi i^\nu H_\nu^{(1)}(-z) = -\frac{1}{2}i\pi(-i)^\nu H_\nu^{(2)}(z)$
$(H_\nu^{(1)}(z))^\star = H_\nu^{(2)}(z^\star), \; (H_\nu^{(2)}(z))^\star = H_\nu^{(1)}(z^\star).$

Thus
$(K_\nu(iz))^\star = \left(\frac{1}{2}i\pi i^\nu H_\nu^{(1)}(-z)\right)^\star = -\frac{1}{2}i\pi(-i)^\nu(H_\nu^{(1)}(-z))^\star = -\frac{1}{2}i\pi(-i)^\nu H_\nu^{(2)}(-z^\star) = K_\nu(i(-z)^\star) = K_\nu((iz)^\star).$ ■

Lemma 6 *The conjugate kernel is given by $\widetilde{E_a(x)} = E_{-a^\star}(-x)$.*

Proof: We have

$E_a(x) =$

$$e^{[ax]_0}\left\{\frac{1}{(2\pi)^{\frac{n}{2}}}\sum_{j=1}^{n}\frac{x_je_j}{|x|^n}(ia_0|x|)^{\frac{n}{2}}K_{\frac{n}{2}}(ia_0|x|) + \frac{ia_0}{(2\pi)^{\frac{n}{2}}}\frac{1}{|x|^{n-2}}(ia_0|x|)^{\frac{n}{2}-1}K_{\frac{n}{2}-1}(ia_0|x|)\right\}$$

From Lemma 5 we get

$$(K_{\frac{n}{2}-1}(ia_0|x|))^\star = K_{\frac{n}{2}-1}(-ia_0^\star|x|) \text{ and } (K_{\frac{n}{2}}(ia_0|x|))^\star = K_{\frac{n}{2}}(-ia_0^\star|x|).$$

and thus

$$\begin{aligned}\widetilde{E_a(x)} = e^{[-a^\star(-x)]_0}\Bigg\{ &\frac{1}{(2\pi)^{\frac{n}{2}}}\sum_{j=1}^n \frac{-x_j e_j}{|x|^n}(-ia_0^\star|x|)^{\frac{n}{2}} K_{\frac{n}{2}}(-ia_0^\star|x|)+\\ &+\frac{-ia_0^\star}{(2\pi)^{\frac{n}{2}}}\frac{1}{|x|^{n-2}}(-ia_0^\star|x|)^{\frac{n}{2}-1}K_{\frac{n}{2}-1}(-ia_0^\star|x|)\Bigg\} = E_{-a^\star}(-x)\end{aligned}$$

■

Theorem 11 *We have*

$$(S_a^* v)(y) = n(y)(S_{-a^\star}\ nv)(y).$$

In particular for $a = 0$ we have $S^* = nSn$ and if $\Gamma = \mathbb{R}^{n-1}$ we have $S^* = S$.

Proof: We consider

$$\begin{aligned}(S_a u, v) &= \int_\Gamma \left(\left(2\int_\Gamma E_a(x-y)n(y)u(y)d\Gamma_y\right)^{\sim}\right) v(x)d\Gamma_x\\ &= \int_\Gamma 2\int_\Gamma \widetilde{u(y)}\ \overline{n(y)}\ \widetilde{E_a}(x-y)d\Gamma_y v(x)d\Gamma_x\\ &= \int_\Gamma \widetilde{u(y)}\ \overline{n(y)}\ 2\int_\Gamma \widetilde{E_a}(x-y)v(x)d\Gamma_x d\Gamma_y\\ &= \int_\Gamma \widetilde{u(y)}\ \overline{n(y)}\ 2\int_\Gamma \widetilde{E_a}(x-y)\ \ \overline{n(x)}n(x)v(x)d\Gamma_x d\Gamma_y\\ &= \int_\Gamma \widetilde{u(y)}\left\{-2\ n(y)\int_\Gamma \widetilde{E_a}(x-y)v(x)d\Gamma_x\right\}d\Gamma_y = (u, S_a^* v).\end{aligned}$$

Hence,

$$\begin{aligned}(S_a^* v)(y) &= -2n(y)\int_\Gamma \widetilde{E_a}(x-y)\ v(x)d\Gamma_x\\ &= -2n(y)\int_\Gamma E_{-a^\star}(y-x)\ n(x)(-n(x))v(x)d\Gamma_x = n(y)(S_{-a^\star}\ nv)(y).\end{aligned}$$

If $a = 0$ then $S = nS^*n$. If moreover $\Gamma = \mathbb{R}^{n-1}$ then $n = e_n$ and $\overline{E(x-y)}e_n = -e_n\overline{E(x-y)}$ and thus

$$(S^* v)(y) = -2e_n\int_\Gamma \overline{(E(x-y))}(-e_n^2)v(x)d\Gamma_x$$

$$= -2e_n^2\int_\Gamma E(y-x)e_n v(x)d\Gamma_x = (Sv)(y). \quad ■$$

Using this we get $P_a^* = \frac{1}{2}(I + S_a^*)$ and $Q_a^* = \frac{1}{2}(I - S_a^*)$.

10 Dense subsets related to the orthogonal decomposition

In this section we want to describe dense subsets of $\text{im } P_a$ and $\text{im } Q_a$ in $L_{2,\mathcal{C}}$. Because S_a^* can be expressed in terms of $S_{-a^\star}$ we find the following useful relations.

Lemma 7 *Let $v \in L_{2,\mathcal{C}}(\Gamma)$. Then*

$$v \in \ker P_a^* \iff nv \in \text{ im } P_{-a^*} \text{ and } v \in \ker Q_a^* \iff nv \in \text{ im } Q_{-a^*}.$$

Proof: $P_a^* v = \frac{1}{2}(I + S_a^*)v = 0 \iff S_a^* v = -v = nS_{-a^\star}nv$ or $nv = S_{-a^\star}nv$ and this means $nv \in \text{ im } P_{-a^*}$. The other relation can be proved analogously. ■

Theorem 12 *Let Γ_a be a smooth Liapunov surface such that $\Gamma_a \subset \mathbb{R}^n \backslash \overline{G}$ and $\text{dist}(G, \Gamma_a) > 0$ and let $\{x_n^{(e)}\}_{n=1}^\infty$ be a dense subset of Γ_a. Then $\{E_a(x - x_n^{(e)})\}_{n=0}^\infty$ is a dense subset of $\text{ im } P_a \cap L_{2,\mathcal{C}}(\Gamma)$.*

Proof: First, if $v \in \ker P_a^*$ then $nv \in \text{ im } P_a$ and thus

$$\begin{aligned}(E_a(\,.\,- x_n^{(e)}), v) &= \int_\Gamma \widetilde{E_a}(y - x_n^{(e)})v(y)d\Gamma = \int_\Gamma E_{-a^\star}(x_n^{(e)} - y)n(y)(-n(y))v(y)d\Gamma \\ &= -\left(F_{-a^\star}(nv)\right)(x_n^{(e)}) = 0 \qquad \forall n \in \mathbb{N}.\end{aligned}$$

On the other hand, if

$$(E_a(\,.\,- x_n^{(e)}), v) = \int_\Gamma \widetilde{E_a}(y - x_n^{(e)})n(y)\{-n(y)v(y)\}d\Gamma$$
$$= -\left(F_{-a^\star}(nv)\right)(x_n^{(e)}) = 0 \; \forall n \in \mathbb{N},$$

then $F_{-a^\star}(nv) = 0$ on Γ_a and because of Lusin‘s theorem we have $F_{-a^\star}(nv) = 0$ on $\mathbb{R}^n \backslash \overline{G}$ and $nv \in \text{ im } P_{-a^*}$ and this is equivalent to $v \in \ker P_a^*$. Thus

$$(E_a(\,.\,- x_n^{(e)}), v) = 0 \qquad \forall n \in \mathbb{N}$$

implies $v = 0$ in $\text{ im } P_a$ and so $\{E_a(x - x_n^{(e)})\}_n^\infty$ is a dense subset in $\text{ im } P_a$. ■

In an analogous way we can prove the following theorem

Theorem 13 *Let Γ_i be a smooth Liapunov-surface such that $\Gamma_i \subset G$ and $\text{dist}(G, \Gamma_i) > 0$ and let $\{x_n^{(i)}\}_{n=1}^\infty$ be a dense subset of Γ_i. Then $\{E_a(x - x_n^{(i)})\}_{n=0}^\infty$ is a dense subset of $\text{ im } Q_a \cap L_{2,\mathcal{C}}(\Gamma)$.*

Combining both theorems we can state

Theorem 14 *Let $\Gamma, \Gamma_i, \Gamma_a$ and $x_n^{(i)}$ and let $x_n^{(e)}$ be as in Theorems 12 and 13. Then the set $\{E_a(x - x_n^{(i)})\}_{n=0}^\infty \cup \{E_a(x - x_n^{(e)})\}_{n=0}^\infty$ is dense in $L_{2,\mathcal{C}}(\Gamma)$.*

Proof: An arbitrary element $u \in L_{2,\mathcal{C}}(\Gamma)$ may be written as

$$u = \frac{1}{2}(I + S_a + I - S_a)u = P_a u + Q_a u.$$

■

11 Application to Maxwell's equations

In the physical setting of the problems we follow here [5].
Maxwell's equations are the fundamental equations of electromagnetism. Electromagnetic phenomena in vacuo are described with the help of two functions $\mathbf{E}$ and $\mathbf{B}$ defined on the hole space $\mathbb{R}^3_x \times \mathbb{R}_t$ with vector values in $\mathbb{R}^3$-called respectively the electric field and the magnetic induction.
These functions $\mathbf{E}$ and $\mathbf{B}$ are linked with two functions ρ and $\mathbf{j}$ defined likewise on $\mathbb{R}^3_x \times \mathbb{R}_t$, with $\rho(x,t) \in \mathbb{R}$ and $\mathbf{j}(x,t) \in \mathbb{R}^3$-called respectively charge density and current density - by the equations, called Maxwell's equations:

$$\begin{cases} -\dfrac{\partial \mathbf{E}}{\partial t} + \text{rot } \mathbf{B} - \mathbf{j} = 0 & \text{the Maxwell-Ampère law,} \\ \text{div } \mathbf{E} - \rho = 0 & \text{Gauss' electric law,} \\ \dfrac{\partial \mathbf{B}}{\partial t} + \text{rot } \mathbf{E} = 0 & \text{the Maxwell-Faraday law,} \\ \text{div } \mathbf{B} = 0 & \text{Gauss' magnetic law,} \end{cases}$$

with the usual notation (for $\mathbf{E} = (E_1, E_2, E_3),\ x = (x_1, x_2, x_3)$)

$$\begin{cases} \text{div } \mathbf{E} = \displaystyle\sum_{i=1}^{3} \frac{\partial E_i}{\partial x_i}, \\ \text{rot } \mathbf{E} = \left(\dfrac{\partial E_3}{\partial x_2} - \dfrac{\partial E_2}{\partial x_3}, \dfrac{\partial E_1}{\partial x_3} - \dfrac{\partial E_3}{\partial x_1}, \dfrac{\partial E_2}{\partial x_1} - \dfrac{\partial E_1}{\partial x_2} \right) \end{cases}$$

In many problems concerning Maxwell's microscopic equations (in vacuo and in the whole space $\mathbb{R}^3_x \times \mathbb{R}_t$) one uses, instead of the functions $\mathbf{E}$ and $\mathbf{B}$, the two functions:

$$\begin{cases} (x,t) \to \mathbf{A}(x,t) \in \mathbb{R}^3 \text{ called "the vector potential",} \\ \text{and} \\ (x,t) \to V(x,t) \in \mathbb{R} \text{ called "the scalar potential",} \end{cases}$$

which are related to $\mathbf{E}$ and $\mathbf{B}$ by

$$\begin{cases} \mathbf{B} = \text{rot } \mathbf{A}, \\ \mathbf{E} = -\text{grad } V - \dfrac{\partial \mathbf{A}}{\partial t}. \end{cases} \tag{1}$$

Substituting the expressions into Maxwell's equations, we obtain the inhomogeneous linear system:

$$\begin{cases} \dfrac{\partial^2 \mathbf{A}}{\partial t^2} - \triangle A + \text{grad } \left(\text{div } \mathbf{A} + \dfrac{\partial V}{\partial t} \right) = \mathbf{j}, \\ -\triangle V - \dfrac{\partial}{\partial t}(\text{div } \mathbf{A}) = \rho. \end{cases} \tag{2}$$

We observe that the functions $\mathbf{A}$ and V are not defined in a unique manner by (1) starting from $\mathbf{E}$ and $\mathbf{B}$: if $\mathbf{A}$ and V satisfy (1),then for any arbitrary function u of x and t, $\mathbf{A}'$ and V' defined by:

$$\begin{cases} \mathbf{A}' = \mathbf{A} + \text{grad } u, \\ V' = V - \dfrac{\partial u}{\partial t} \end{cases} \tag{3}$$

also satisfy (1). The transformation $(\mathbf{A}, V) \to (\mathbf{A}', V')$ given by (3) is called a gauge transformation. As a result of (3), we have (always in $\mathbb{R}^3_x \times \mathbb{R}_t$)

$$\text{div } \mathbf{A}' + \frac{\partial V'}{\partial t} = \text{div } \mathbf{A} + \frac{\partial V}{\partial t} + \triangle u - \frac{\partial^2 u}{\partial t^2} \tag{4}$$

Taking for u a solution of the equation

$$\triangle u - \frac{\partial^2 u}{\partial t^2} = -\left(\text{div } \mathbf{A} + \frac{\partial V}{\partial t}\right)$$

(where $\mathbf{A}$ and V are supposed to be known), we see that it is possible to choose a pair $(\mathbf{A}_L, V_L)$ such that

$$\text{div } \mathbf{A}_L + \frac{\partial V_L}{\partial t} = 0. \tag{5}$$

This relation is called the Lorentz condition. With this choise, equations (2) can be written:

$$\begin{cases} \dfrac{\partial^2 \mathbf{A}_L}{\partial t^2} - \triangle \mathbf{A}_L = \mathbf{j}, \\ \dfrac{\partial^2 V_L}{\partial t^2} - \triangle V_L = \rho. \end{cases} \tag{6}$$

Note that (4) with (6) does not determine a unique pair $(\ bA_L, V_L)$ when $\mathbf{j}$ and ρ are known.

As an application of the Hilbert problem we want to consider stationary problems. The expression ”stationary problems” demands a precise definition. We understand by that the search for solutions of Maxwell's equations in the whole space which are of the form

$$\begin{cases} \mathbf{E}(x,t) = \mathbf{E}_0(x) e^{i\omega t} \\ \mathbf{B}(x,t) = \mathbf{B}_0(x) e^{i\omega t} \end{cases} \tag{7}$$

with ω a known non-zero real constant. An electromagnetic wave with such a solution is said to be monochromatic- the electric field then being made up of functions which are periodic in time (but obviously not every periodic solution of Maxwell's equations is of this form). The constant ω is called the pulsation of the electromagnetic field, and $\frac{\omega}{2\pi}$ its frequency. We see the charge density ρ and current density $\mathbf{j}$ can be represented by means of

$$\begin{cases} \rho(x,t) = \rho_0(x) e^{i\omega t} \\ \mathbf{j}(x,t) = \mathbf{j}_0(x) e^{i\omega t}. \end{cases}$$

and also the vector and scalar potential:

$$\begin{cases} V(x,t) = V_0(x)e^{i\omega t} \\ \mathbf{A}(x,t) = \mathbf{A}_0(x)e^{i\omega t}. \end{cases}$$

Using these relations we obtain from (1) the equations

$$\begin{cases} \mathbf{B}_0(x) = \text{rot } \mathbf{A}_0(x) \ , \\ \mathbf{E}_0(x) = - \text{ grad } V_0(x) - i\omega\mathbf{A}_0(x). \end{cases}$$

We want to put the relations into Clifford algebra language. For this purpose we consider Du with the Dirac operator D in $\mathcal{C}_{0,3}$ and u a paravector $u = u_0e_0 + \sum_{i=1}^{3} u_ie_i = u_0e_0 + \underline{u}$. We get

$$Du = \begin{pmatrix} \text{- div } \underline{u} \\ \text{grad } u_0 \\ \text{rot } \underline{u} \end{pmatrix} \text{ and } (D+\alpha_0)u = \begin{pmatrix} \text{- div } \underline{u} + \alpha_0 u_0 \\ \text{grad } u_0 + \alpha_0\underline{u} \\ \text{rot } \underline{u} \end{pmatrix}.$$

Thus we interprete the scalar and the vector potential as a special paravector $F(x) = iV_0(x) + \mathbf{A}_0(x)$ and the electric field and the magnetic induction as the element $U(x) = -i\mathbf{E}_0(x) + \mathbf{B}_0(x)$ of $\mathcal{C}_{0,3}$ and put

$$U(x) = (D - i\omega)F(x)$$

The scalar part of this equation is zero and represents the Lorentz condition. The pseudoscalar part equals zero on both sides. The rest is easily seen from

$$\begin{pmatrix} 0 \\ -i\mathbf{E}_0(x) \\ \mathbf{B}_0(x) \\ 0 \end{pmatrix} = (D - i\omega) \begin{pmatrix} iV_0(x) \\ \mathbf{A}_0(x) \\ \mathbf{O} \\ 0 \end{pmatrix} = \begin{pmatrix} \text{-div } \mathbf{A}_0(x) + ii\omega iV_0(x) \\ \text{igrad } V_0(x) + ii\omega\mathbf{A}_0(x) \\ \text{rot } \mathbf{A}_0(x) \\ 0 \end{pmatrix} =$$

$$\begin{pmatrix} \text{-div } \mathbf{A}_0(x) - i\omega V_0(x) \\ \text{i(grad } V_0(x) + i\omega\mathbf{A}_0(x)) \\ \text{rot } \mathbf{A}_0(x) \\ 0 \end{pmatrix}.$$

The equations (6) for the scalar and vector potential are moved into

$$(\triangle + \omega^2)F(x) = R(x) \tag{8}$$

where $R(x) = (-i\rho_0(x), -\mathbf{j}_0(x))$. In terms of the Dirac operator (8) is equivalent to

$$(D+\omega)(D-\omega)F(x) = -R(x).$$

If $R(x) = 0$ (this means there is no source) then our solution can be represented as

$$F(x) = \Phi(x) + \Psi(x)$$

where Φ fullfills $(D+\omega)\Phi(x)=0$ in $\mathbb{R}^3_+$ and Ψ fullfills $(D+\omega)\Psi(x)=0$ in $\mathbb{R}^3_-$.

Then the Hilbert problem means to determine a $F(x)$ that fullfills

$$(D+\omega)(D-\omega)F(x)=0$$

and there is a linear relation between Ψ and Φ on $\partial\mathbb{R}^3_+ = \partial\mathbb{R}^3_- = \mathbb{R}^2$. That is

$$\Phi(x) = H(x)\Psi(x) + h(x).$$

Acknowledgement

The author is indebted to Professor R. Delanghe for discussions and suggestions.

References

[1] H. Bateman, A. Erdèlyi, *Higher Transcendental Functions*, vol I -III, Mc Graw-Hill Company, INC, New York, 1953.

[2] S. Bernstein, *Fundamental solutions for Dirac-type operators*, Banach Center Symposium on Generalizations of Complex Analysis and Their Applications in Physiscs, 1994, Banach Center Publ., Warsaw, in print.

[3] S. Bernstein, *On the index of Clifford-valued singular integral operators and the left-linear Riemann problem*, to appear.

[4] F. Brackx, R. Delanghe, F. Sommen, *Clifford Analysis*, Pitman Advanced Publishing Program, Boston-London-Melbourne, 1982.

[5] R. Dautray, J.-L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology*, vol. 1, Springer-Verlag, Berlin-Heidelberg-New York, 1990.

[6] R. Delanghe, F. Sommen, V. Souček, *Clifford Algebra and Spinor-Valued Functions*, Mathematics and Its Applications 53, Kluwer Academic Publishers, Dordrecht, Boston, London, 1992.

[7] J. E. Gilbert, M. A. M. Murray, *Clifford algebras and Dirac operators in harmonic analysis*, Cambridge studies in advanced mathematics 26, Cambridge University Press, Cambridge New York, 1991.

[8] K. Gürlebeck, W. Sprößig, *Quaternionic Analysis and Elliptic Boundary Value Problems*, ISNM 89, Birkhäuser Verlag, Basel, Boston, Berlin, 1990.

[9] V.V. Kravchenko, *Integral representations for biquaternionic hyperholomorphic functions and their applications*, PhD thesis, State University Rostov, 1993, (Russian).

[10] R. Kress, *Linear Integral Equations*, Applied Mathematical Science 82, Springer-Verlag, Berlin Heidelberg New York, 1989.

[11] S.G. Mikhlin, S. Prößdorf, *Singular Integral Operators*, Akademie- Verlag Berlin, 1986.

[12] E. Obolashvili, *Space generalized holomorphic vectors*, Diff. Uravn. T.X1,1, (1975), 108-115, Minsk (Russian).

[13] E. Obolashvili, *Effective solutions of some boundary value problems in two and three dimensional cases*, Functional analytic methods in complex analysis and applications to PDE, 1988, Trieste, 149-172.

[14] N. Ortner, *Regularisierte Faltung von Distributionen. Teil 2: Eine Tabelle von Fundamentallösungen*, Journal of Applied Mathematics and Physics (ZAMP), Vol. 31, 1980, 133-155.

[15] R.T. Seeley, *The index of elliptic systems of singular integral operators*, J. Math. Anal. Appl. **7**:2, 289-309(1963).

[16] M.V. Shapiro, *On some boundary value problems for functions with values in a Clifford algebra*, MaT. Bec. 40(1988).

[17] M.V. Shapiro, N.L. Vasilevski, *Quaternionic ψ-hyperholomorphic functions, singular operators with quaternionic Cauchy kernel and analogies of the Riemann boundary value problem* I and II, Complex Variables 27 (1995) 17-27 and 67-96.

[18] I. Stern, *Randwertaufgaben für verallgemeinerte Cauchy-Riemann-Systeme im Raum*, Dissertation, Martin-Luther-Universität Halle-Wittenberg, 1989.

[19] R. Temam, *Navier-Stokes equations*, (Studies in math. and its appl.; v. 2), North-Holland, Amsterdam New York Oxford, 1985.

[20] Z. Xu, *A Function Theory for the Operator $D - \lambda$*, Complex Variables, Theory App., 1991, Vol. 16, pp. 27-42.

[21] Z. Xu, *Helmholtz equations and boundary value problems*, in Partial differential equations with complex analysis (H. Begehr and A. Jeffrey, Eds.), Pitman Research Notes in Mathematics Series 262 (1992) 204-214.

TU Bergakademie Freiberg
Institut für Angewandte Mathematik I
D-09596 Freiberg