Covers of $\mathrm{PG}(3, q)$ and of finite generalized quadrangles

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Abstract

This article studies covers in $\mathrm{PG}(3, q)$ and in generalized quadrangles. The excess of a cover is defined to be the difference between the number of lines in the cover and the number of lines in a spread. In contrast with the theory of partial spreads which tells us that large partial spreads can be extended to spreads, in $\mathrm{PG}(3, q)$ and in some generalized quadrangles, there exist minimal covers with small excess. For such minimal covers with small excess, we describe the structure of the set of points lying on at least two lines of the cover.

1 Introduction

Let $\Sigma = \mathrm{PG}(3, q)$ be the 3-dimensional projective space over the finite field $\mathbb{F}_q$ of order q. A *0-cover* of Σ is a mapping α from the set of lines of Σ into $\mathbb{Z}$ associating a weight α_L to each line L such that for each point of Σ, the sum of the weights of the lines passing through that point is at least zero. A *minimal* 0-cover is a 0-cover α such that the weight of every line of Σ is minimal. This means that every line of Σ contains at least one point with weight zero. The *excess* of a 0-cover is the sum of the weights of the lines of Σ.

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A *partial spread* is a set of skew lines of Σ. A partial spread is called *maximal* when it is not contained in a larger partial spread. A *spread* of Σ is a set of q^2+1 lines of Σ which form a partition of the points of Σ. A *cover* of Σ is a set of lines of Σ such that each point of Σ belongs to at least one line of the cover. A cover is called *minimal* when no proper subset of it is still a cover. Equivalently, a cover $\mathcal{M}$ is minimal when each line of $\mathcal{M}$ contains a point on no other line of $\mathcal{M}$. The *excess* of a cover is equal to the number of lines in the cover minus q^2+1. A point of Σ is called a *multiple point* of a cover when it belongs to at least two lines of the cover.

A *blocking set* K of the projective plane $\mathrm{PG}(2,q)$, defined over $\mathbb{F}_q$, is a set of points such that each line of $\mathrm{PG}(2,q)$ contains at least one point of K. A blocking set K containing a line of $\mathrm{PG}(2,q)$ is called a *trivial* blocking set. When K does not contain a line, it is called a *non-trivial* blocking set.

Let $\mathcal{L}$ be a collection of lines of $\mathrm{PG}(3,q)$, where each line is accorded a non-negative integer called its *weight*. The set of points which lie on at least one element of $\mathcal{L}$ is called the *sum* of the lines $\mathcal{L}$. Further, the *weight* of a point p in the sum of lines $\mathcal{L}$ is the sum of the weights of the lines of $\mathcal{L}$ passing through p.

A generalized quadrangle $\mathrm{GQ}(s,t)$, with parameters (s,t), $s \geq 1$, $t \geq 1$, is an incidence structure $\mathcal{S} = (P, B, \mathrm{I})$ in which P and B are disjoint, non-empty sets of objects, called respectively *points* and *lines*, and for which I is a symmetric point-line incidence relation satisfying the following axioms:

(i) each point is incident with $1+t$ lines and two distinct points are incident with at most one line;
(ii) each line is incident with $1+s$ points and two distinct lines are incident with at most one point;
(iii) if x is a point and L is a line not incident with x, then there is a unique pair $(y, M) \in P \times B$ for which $x \mathrm{I} M \mathrm{I} y \mathrm{I} L$.

A *spread* of $\mathcal{S}$ is a set of lines that partitions the point set, that is, a set of $st+1$ pairwise non-concurrent lines. A *(minimal) cover* is defined as above. If $\mathcal{M}$ is a minimal cover with $st+1+r$ lines, we say $\mathcal{M}$ has *excess* r.

Partial spreads have already been studied in detail. With respect to maximality, the following results are known:
(i) in $\mathrm{PG}(3,q)$, a partial spread containing more than $q^2+1-\sqrt{2q}$ lines is contained in a spread of $\mathrm{PG}(3,q)$ [5];
(ii) in $\mathrm{PG}(3,q)$, q non-square, a partial spread containing $q^2+1-\delta$ lines, for which $\delta > 0$ and $8\delta^3 - 18\delta^2 + 8\delta + 4 < 3q^2$, is contained in a spread of $\mathrm{PG}(3,q)$ [12];
(iii) in a $\mathrm{GQ}(s,t)$, a partial spread containing more than $st - s/t$ lines is contained in a spread [16, 2.7.1].

We address the "dual" problem of 0-covers and covers. For instance, for which values of r does there exist a minimal 0-cover of excess r?

The following fundamental examples show that $\mathrm{PG}(3,q)$ and all $\mathrm{GQ}(s,t)$ have minimal covers with excess respectively q and $t-1$. In $\mathrm{PG}(3,q)$, consider the star of lines through a fixed point; these lines form a minimal cover of excess q. In $\mathcal{S} = \mathrm{GQ}(s,t)$, let L be a line of $\mathcal{S}$ and put $\mathcal{M} = L^\perp \setminus \{L\}$. Then $\mathcal{M}$ is a minimal cover with $|\mathcal{M}| = (1+s)t = st+1+(t-1)$.

In fact, we will show that in $\mathrm{PG}(3,q)$ there exist minimal covers with excess r for all r with $0 \le r \le q$. A similar result holds for all $\mathrm{GQ}(q,q^2)$ that are point-line duals of those arising from flocks of quadratic cones. In these generalized quadrangles, there exist minimal covers with excess r for all r with $0 \le r \le q^2-1$. However, we also give an example of a generalized quadrangle having no minimal cover with excess 1.

We also remark that in $\mathcal{S}$=$\mathrm{GQ}(s,t)$, the point-line dual of a cover is a *blocking set*, that is, a set $\mathcal{K}$ of points of $\mathcal{S}$ such that each line of $\mathcal{S}$ is incident with some point of $\mathcal{K}$. A *blocking set with excess* r is a blocking set of cardinality $st+1+r$.

Hence our results for covers in generalized quadrangles of order (s,t) can be translated into results on blocking sets of the dual generalized quadrangles.

2 Flock generalized quadrangles

2.1 Flock generalized quadrangles

To describe a flock GQ, we proceed in the following way.

A q-clan $\mathcal{C} = \{A_t || t \in \mathbb{F}_q\}$ is a set of q matrices $A_t = \begin{pmatrix} x_t & y_t \\ 0 & z_t \end{pmatrix}, t \in \mathbb{F}_q$, such that, whenever $s \neq t$, the matrix $A_s - A_t$ is anisotropic, which means that $\bar{x}(A_s - A_t)\bar{x}^T = 0$ has only the trivial solution $\bar{x} = (0,0)$.

Starting with a q-clan $\mathcal{C}$, a $\mathrm{GQ}(q^2,q)$ $\mathcal{S}(\mathcal{C})$ is constructed as a group coset geometry. Let $\mathcal{K}$ denote the group consisting of the set $\mathcal{K} = \{(\alpha, c, \beta) \in \mathbb{F}_q^2 \times \mathbb{F}_q \times \mathbb{F}_q^2 || \alpha, \beta \in \mathbb{F}_q^2, c \in \mathbb{F}_q\}$, together with the binary operation $(\alpha,c,\beta)(\alpha',c',\beta') = (\alpha+\alpha', c+c'+\beta\cdot\alpha', \beta+\beta')$. (Here $\beta\cdot\alpha'$ is the usual dot product of vectors in $\mathbb{F}_q^2$.)

Then define $q+1$ subgroups of $\mathcal{K}$ having order q^2: $A(\infty) = \{(\bar{0},0,\beta) \in \mathcal{K} || \beta \in \mathbb{F}_q^2\}$; $A(t) = \{(\alpha, \alpha A_t \alpha^T, \alpha K_t) \in \mathcal{K} || \alpha \in \mathbb{F}_q^2\}, t \in \mathbb{F}_q$. Here $K_t = A_t + A_t^T$, $t \in \mathbb{F}_q$.

Put $\mathcal{J} = \{A(t) || t \in \mathbb{F}_q \cup \{\infty\}\}$, and for each $A \in \mathcal{J}$, define a subgroup A^* containing A in the following way: $A^*(\infty) = \{(\bar{0}, c, \beta) \in \mathcal{K} || c \in \mathbb{F}_q, \beta \in \mathbb{F}_q^2\}$; $A^*(t) = \{(\alpha, c, \alpha K_t) \in \mathcal{K} || \alpha \in \mathbb{F}_q^2, c \in \mathbb{F}_q\}$.

Put $\mathcal{J}^* = \{A^* || A \in \mathcal{J}\}$. Then $\mathcal{J}$ is a *4-gonal family for* $\mathcal{K}$, that is, $\mathcal{J}$ and $\mathcal{J}^*$ satisfy the properties of W.M. Kantor [10] with $s = q^2, t = q$, so that a GQ $\mathcal{S}(\mathcal{C}) = (P, B, \mathrm{I})$ may be constructed with the following points, lines and incidence.

The points are: (i) (∞), (ii) right cosets $A^*(t)g, g \in \mathcal{K}, t \in \mathbb{F}_q \cup \{\infty\}$, (iii) elements $g \in \mathcal{K}$. The lines are: (a) $[A(t)], t \in \mathbb{F}_q \cup \{\infty\}$ and (b) right cosets $A(t)g, g \in \mathcal{K}, t \in \mathbb{F}_q \cup \{\infty\}$. The incidence: (∞) is incident with each line of type (a), the point $A^*(t)g$ is incident with $[A(t)]$ and with each line $A(t)h$ of type (b) contained in it, the point g of type (iii) is incident with each line $A(t)g$ of type (b) containing it. There are no further incidences.

2.2 The Knarr construction

For q odd, N. Knarr [11] has given the following geometrical construction of the flock generalized quadrangles starting with a BLT-set in $W(3,q)$.

Consider in $\mathrm{PG}(3,q)$, q odd, the symplectic geometry $W(3,q)$. A *BLT-set* of lines of $W(3,q)$ is a set $\mathcal{S}'$ of $q+1$ totally singular lines of $W(3,q)$, no two concurrent, such that each totally singular line of $W(3,q)$ not in $\mathcal{S}'$ is concurrent with exactly 0 or 2 lines of $\mathcal{S}'$.

To construct the flock GQ, start with a symplectic polarity ζ of $\mathrm{PG}(5,q)$. Let $p \in \mathrm{PG}(5,q)$ and let $\mathrm{PG}(3,q)$ be a 3-dimensional subspace of $\mathrm{PG}(5,q)$ for which $p \notin \mathrm{PG}(3,q) \subseteq p^\zeta$. In $\mathrm{PG}(3,q)$, ζ induces a symplectic polarity ζ', and hence a symplectic geometry $W(3,q)$. Let $V = \{L_0, \ldots, L_q\}$ be a BLT-set of lines of $W(3,q)$. Construct a geometry $\mathcal{S}(V) = (P, B, \mathrm{I})$ in the following way.

The points are: (i) p, (ii) lines of $\mathrm{PG}(5,q)$ not containing p but contained in one of the planes $\pi_t = \langle p, L_t \rangle$, $0 \leq t \leq q$, (iii) points of $\mathrm{PG}(5,q) \setminus p^\zeta$. The lines are: (a) planes $\pi_t = \langle p, L_t \rangle, 0 \leq t \leq q$, and (b) totally singular planes of ζ not contained in p^ζ and meeting some π_t in a line (not through p). The incidence relation I is the natural incidence inherited from $\mathrm{PG}(5,q)$.

Knarr shows that $\mathcal{S}(V)$ is a flock generalized quadrangle, where the flock is associated with the BLT-set via [2].

3 Minimal covers in $\mathrm{PG}(3,q)$

We have already seen that $\mathrm{PG}(3,q)$ admits minimal covers of excess 0, that is, spreads, and of excess q. In this section we construct further examples of minimal covers of $\mathrm{PG}(3,q)$.

Theorem 1 *Let $\mathcal{S}$ be a spread of $\mathrm{PG}(3,q)$ and let $\mathcal{R} = \cup_{i=1}^m R_i$ be the union of m reguli contained in $\mathcal{S}$, with $R_i \not\subseteq \cup_{j\neq i} R_j$, for all $i = 1, \ldots, m$. For $i = 1, \ldots, m$, let R_i^{opp} be the opposite regulus to R_i. If $\mathcal{R}$ contains $m(q+1) - r$ lines of $\mathcal{S}$, then $\mathcal{M} = (\mathcal{S} \setminus \mathcal{R}) \cup_{i=1}^m R_i^{\mathrm{opp}}$ is a minimal cover of $\mathrm{PG}(3,q)$ of excess r.*

Proof. It is immediate that $\mathcal{M}$ is a cover, and that no line of $\mathcal{M} \cap \mathcal{S}$ can be deleted to obtain a smaller cover. Suppose that $\mathcal{M} \setminus \{L\}$ is a cover, where L is a line of R_i^{opp} for some i. Then each point of L lies on a line of a regulus R_j^{opp}, for some $j \neq i$. Thus $R_i \subseteq \cup_{j\neq i} R_j$; a contradiction. Since all opposite reguli R_i^{opp} are pairwise disjoint, $|\mathcal{M}| = q^2 + 1 - (m(q+1) - r) + m(q+1)$; so $\mathcal{M}$ has excess r. ■

If $\mathcal{S}$ is the regular spread, then since every three lines of $\mathcal{S}$ lie in a regulus contained in $\mathcal{S}$, it is possible to construct a great variety of minimal covers of this kind.

Note also that the condition of Theorem 1 is trivially fulfilled for $m < (q+3)/2$.

Suppose that $\mathcal{S}$ is a spread comprising q reguli sharing exactly one line (there exist projectively distinct classes of such spreads, corresponding to flocks of quadratic cones [8, Theorem 2.2]). Choosing $\mathcal{R}$ to be the union of $r+1$ of these reguli gives a minimal cover of excess r for each $0 \leq r \leq q-1$.

Let $\mathcal{S}$ be the regular spread in $\mathrm{PG}(3,q)$ for q odd, and let $\mathcal{R}$ be the union of three reguli in $\mathcal{S}$, pairwise meeting in a line but with no common line (such a set $\mathcal{R}$ exists; under the Klein correspondence it is the union of three conics, pairwise tangent in three distinct points, on a 3-dimensional elliptic quadric contained in the Klein quadric). This provides a different example of a minimal cover of $\mathrm{PG}(3,q)$, q odd, of excess 3.

Let $\mathcal{S}$ be a spread comprising $q+1$ reguli which share exactly two lines (there exist projectively distinct classes of such spreads, corresponding to flocks of hyperbolic quadrics [1, 17, 18]). Choosing $\mathcal{R}$ to be the union of m of these reguli gives a minimal cover of excess $2(m-1)$ for each $1 \leq m \leq q+1$, that is, excesses $0, 2, 4, \ldots, 2q$ occur.

All the examples of minimal covers constructed in this section have the property that the collection of multiple points forms a sum of lines of PG$(3, q)$. In Section 4 we show that this is a feature of minimal covers of small excess.

4 Minimal 0-covers in PG$(3, q)$

In this section, let α be a minimal 0-cover of excess r. Let $\{p_1, \ldots, p_{q^3+q^2+q+1}\}$ and $\{\pi_1, \ldots, \pi_{q^3+q^2+q+1}\}$ be respectively the point set and plane set of PG$(3, q)$.

Definition 1 *The* excess a_i *of a point* p_i*, also called ex(*p_i*), is defined to be the sum of the weights of the lines of* α *passing through that point.*

The excess b_i *of a plane* π_i *is the sum of the excesses of the points of that plane.*

The excess c_L *of a line* L *is the sum of the excesses of the points of the line.*

The excess of PG$(3, q)$ *is the sum of the excesses of all points of PG*$(3, q)$*.*

Lemma 1 *(i) The excess of PG*$(3, q)$ *is* $r(q+1)$*;*
(ii) The sum of the excesses of all the points of each plane π_i *is equal to* $r \pmod q$*.*

Proof. (i) $\sum_{i=1}^{q^3+q^2+q+1} a_i = \sum_{p_i \in PG(3,q)} \sum_{p_i \in L} \alpha_L = \sum_L \sum_{p_i \in L} \alpha_L = \sum_L (q+1)\alpha_L = (q+1)r$. To prove (ii), let π be a plane. Let $R = \sum_{p \in \pi} ex(p)$. Then each line L contained in π contributes $(q+1)\alpha_L$ to R, while each other line contributes α_L to R. Hence $R \equiv r \pmod q$. ■

Theorem 2 *A minimal 0-cover in* PG$(3, q)$ *has excess 0 or at least* ϵ *where* $q + \epsilon$ *is the size of the smallest non-trivial blocking sets in* PG$(2, q)$*.*

Proof. Let the excess r of the 0-cover satisfy $0 < r < \epsilon$.

In PG$(2, q)$, q odd, there exists a non-trivial blocking set of size $3(q+1)/2$ and in PG$(2, q)$, q even, there exists a non-trivial blocking set of size $3q/2 + 1$ [9]. So we can assume that $r < (q+3)/2$.

We first note that if L is a line of excess 0, then all planes π through L have excess r. Namely, $r(q+1) = \sum_{i=1}^{q+1} b_i$ where b_i is the excess of the plane π_i containing L. By Lemma 1, we have $b_i = r + l_i q$. So $r(q+1) = r(q+1) + q\sum_{i=1}^{q+1} l_i$. Hence, $\sum_{i=1}^{q+1} l_i = 0$. As $l_i \geq 0$, we have $l_i = 0$ and so $b_i = r, i = 1, \ldots, q+1$.

Case 1. There is a point p of excess 1.

Then it is possible to find a line L through p whose excess is also equal to one. Indeed each point has non-negative excess and the excess of PG$(3, q)$ is equal to $r(q+1)$. But $r(q+1) - 1$ is smaller than $q^2 + q + 1$ which is the total number of lines through p.

Through this line L, there is exactly one plane π of excess $r + q$. Copying the arguments above, $r(q+1) = 1 + \sum_{i=1}^{q+1}(b_i - 1)$ where b_i is the excess of the plane π_i containing L, and where we separately counted the excess of the line L. Continuing as above, the equality $\sum_{i=1}^{q+1} l_i = 1$ is obtained. Hence, there is exactly one plane π_i passing through L with excess $r+q$ since $l_i \geq 0$, $i = 1, \ldots, q+1$.

This is impossible. If π is a plane of excess $r+q$, then, by using the observation made before Case 1, all lines in that plane have positive excess. Hence the points of π with positive excess form a blocking set in π. Since the number of points in π with positive excess is at most $q+r$, and since this is smaller than the size of the smallest non-trivial blocking sets, there is a line M in π whose points all have positive excess.

So it is possible to reduce the weight of that line M and still have a 0-cover. Hence the original 0-cover was not minimal.

Case 2. All points have excess greater than or equal to some value $k > 1$ and there is a point of excess k.

Let p be a point of excess k, then as in Case 1, it is possible to prove that there is a line L passing through p and having excess k.

Using the same arguments as for $k = 1$, it is possible to prove that there is a plane π passing through L of excess at least $r+q$ and at most $r+kq$.

Since through a line of excess 0, there only pass planes of excess r, all lines in π have positive excess. Hence, the points of π with positive excess form a blocking set in π. Since there are at most $(kq+r)/k < q+r$ such points, again there is a line only consisting of points with positive excess. This gives the same contradiction as in Case 1.

This conclusion gives a contradiction since the excess of a point is finite. Thus if the excess of the minimal 0-cover is less than ϵ, then each point has excess 0 and the excess of the minimal 0-cover is 0. ∎

Corollaries

Let $q+r$, $r > 0$, be smaller than the cardinality $q+\epsilon$ of the smallest non-trivial blocking sets in $\mathrm{PG}(2,q)$.

(1) Let $\mathcal{N}$ be a minimal cover of $\mathrm{PG}(3,q)$, with $|\mathcal{N}| = q^2+1+r$, then the multiple points form a sum of lines with the sum of the weights of the lines equal to r.

Moreover, this sum of lines is unique.

(2) Let $\mathcal{M}$ be a partial spread of $\mathrm{PG}(3,q)$, with $|\mathcal{M}| = q^2+1-r$, then $\mathcal{M}$ can be extended in a unique way to a spread of $\mathrm{PG}(3,q)$.

Proof. (1) Give the lines of $\mathcal{N}$ weight one, and all the other lines weight zero. Consider a spread $\mathcal{S}$, and give all the lines of $\mathcal{S}$ weight one, and the lines not in $\mathcal{S}$ weight zero.

Then it is possible to define a 0-cover α of excess r by giving a line of $\mathrm{PG}(3,q)$ a new weight which is the difference of its weight in $\mathcal{N}$ and its weight in $\mathcal{S}$. Since the excess of α is smaller than ϵ, α is not minimal, so there is a line whose weight is not minimal.

Each point on such a line must have positive excess, but the points with positive excess are the points lying on at least two lines of the cover $\mathcal{N}$.

Let L_1 be a line whose weight in α is not minimal. Lower the weight of L_1 a unit. This gives a new 0-cover α_1 with excess $r-1$. Either $r-1=0$ or $0 < r-1 < \epsilon$. In the latter case, we can repeat the arguments and show that there is a line L_2 whose weight is not minimal in α_1. Again, L_2 completely consists of points with positive excess in α since the excess of a point in α is greater than or equal to its excess in α_1. Lower the weight of L_2 a unit; this gives a new 0-cover α_2 with excess $r-2$.

Continuing in this way, a 0-cover α_r of excess 0, and r lines $L_1,\ldots,L_r$, completely consisting of points with positive excess in α, are obtained.

In α_r, there are no points with positive excess. Hence, the points of α with positive excess must all belong to a line $L_1,\ldots,L_r$.

Since the excess of $\mathrm{PG}(3,q)$ with respect to the 0-cover α is equal to $r(q+1)$, the collection of points with positive excess in α forms a sum of these lines $L_1,\ldots,L_r$, where each line is accorded a weight equal to the number of times it appears in $L_1,\ldots,L_r$.

Suppose the collection of points with positive excess can be written in two distinct ways as the sum of lines $L_1,\ldots,L_r$ and $M_1,\ldots,M_r$. Let M_1 be a line different from all lines $L_j, j=1,\ldots,r$, or let M_1 be equal to a line L_j, but with different weights in the corresponding sums. In the latter case, suppose that the weight of M_1 in the second sum is larger than the weight in the first sum. Then the points of M_1 must also belong to other lines L_j in the first sum since their weights in both sums are equal to their excess in α. Since we have less than $q+1$ such lines $L_1,\ldots,L_r$, this is impossible.

(2) Here, it is possible to define a 0-cover by taking a spread $\mathcal{S}$ of $\mathrm{PG}(3,q)$ and define the 0-cover as the difference of $\mathcal{S}$ and of $\mathcal{M}$. Now, the points with positive excess form a sum of lines and the sum of the weights of these lines is r. Since a point with positive excess can only have excess one, these lines are disjoint lines $L_1,\ldots,L_r$. Moreover the points with positive excess are the points of Σ not lying on a line of the partial spread. Hence, the union of $\mathcal{M}$ and $\{L_1,\ldots,L_r\}$ must form a spread of $\mathrm{PG}(3,q)$.

By (1), $\mathcal{M}$ can be extended in a unique way to a spread. ■

Remarks

1. The preceding corollaries were also proved by J. Eisfeld [7].

2. By Blokhuis [3, 4], in $\mathrm{PG}(2,p)$, p prime, $|K|\geq 3(p+1)/2$ for every non-trivial blocking set K, and in $\mathrm{PG}(2,p^{2h+1})$, p prime, $h>0$, $|K|\geq p^{2h+1}+p^{h+1}+1$, for every non-trivial blocking set K.

   In $\mathrm{PG}(2,q)$, q square, by Bruen and Thas [6], $|K|\geq q+\sqrt{q}+1$ for every non-trivial blocking set K.

5 Minimal covers in generalized quadrangles

In this section, we construct minimal covers in generalized quadrangles. Again, it is possible to construct an example of a minimal cover with small excess; an example which occurs in all generalized quadrangles.

Let L be a line of $\mathcal{S}$ and put $\mathcal{M}=L^{\perp}\setminus\{L\}$. Then $\mathcal{M}$ is a minimal cover with $|\mathcal{M}|=(1+s)t=st+1+(t-1)$.

The problem of constructing minimal covers in generalized quadrangles is harder than the corresponding problem in $\mathrm{PG}(3,q)$. There exist generalized quadrangles which do not have spreads. For instance, a $\mathrm{GQ}(t^2,t)$ does not have a spread [16, 1.8.3].

So here arises the fundamental problem of the minimal cardinality of a cover of the generalized quadrangle. A natural problem is to search for a minimal cover of excess one. In contrast to the situation in $\mathrm{PG}(3,q)$, we will give an example of a generalized quadrangle with a spread, but not having a minimal cover of excess one. Nevertheless for the dual flock generalized quadrangles we will be able to give a result comparable to the one obtained for the minimal covers in $\mathrm{PG}(3,q)$. Then, in Section 6, we will concentrate on minimal covers with small excess.

Since minimal covers are minimal blocking sets in the dual generalized quadrangles, we will use both types of objects.

5.1 Examples

(1) Let $\mathcal{S}'$ be a subquadrangle of order q of a GQ $\mathcal{S}$ of order (q^2,q). Each point of $\mathcal{S}\setminus\mathcal{S}'$ is on a unique line of $\mathcal{S}'$ [16, 2.2.1]. Hence the set $\mathcal{M}$ of $(1+q)(1+q^2)$ lines of $\mathcal{S}'$ forms a minimal cover of $\mathcal{S}$ with excess q^2+q.

(2) Suppose $\mathcal{S}$ has a set $T=\{x_0,x_1,\ldots,x_s\}$ of pairwise non-collinear points such that $T^\perp=\{y_0,y_1,\ldots,y_s\}$ also has size $1+s$. For example, this occurs if $\mathcal{S}$ has order s and $\{x_0,x_1\}$ is a regular pair. And it occurs if $\mathcal{S}$ has order (q,q^2) and $\{x_0,x_1,x_2\}$ is a 3-regular triple (cf. [16] for definitions and examples). Each point of $\mathcal{S}$ not in $T\cup T^\perp$ is collinear with exactly two points of $T\cup T^\perp$ [16, 1.4.1]. Let $\mathcal{M}^+$ be the set of all lines incident with at least one point of $T\cup T^\perp$. Then $\mathcal{M}^+$ is a cover, but not a minimal one. Let $\mathcal{M}$ be obtained by removing the $t-s$ lines through x_0 incident with no point of $T^\perp$ and the $t-s$ lines through y_0, $y_0\in T^\perp$, incident with no point of T. Then $|\mathcal{M}|=(1+s)^2+2(1+s)(t-s)-2(t-s)=st+1+s(t-s+2)$. This set $\mathcal{M}$ is a minimal cover with excess $s(t-s+2)$.

(3) Let $\mathcal{S}=\mathrm{GQ}(s,t)$ be a generalized quadrangle with a regular pair of lines $\{L_0,L_1\}$. Let $\{L_0,L_1\}^\perp=\{M_0,\ldots,M_s\}$ and $\{M_0,\ldots,M_s\}^\perp=\{L_0,\ldots,L_s\}$.

Let G be the $(s+1)\times(s+1)$ grid defined by these lines. Then the lines intersecting L_0, but not lying in G, together with the lines $L_1,\ldots,L_s$ form a minimal cover of size $st+t-1$.

For instance, in the unique GQ(3,3) with all lines regular, this gives a minimal cover of size $st+2$.

5.2 Minimal blocking sets in flock generalized quadrangles

5.2.1 The case q is even

Consider a flock generalized quadrangle as described in Section 2.1. Let H be a 2×2 matrix over $\mathbb{F}_q$ having no eigenvalue in $\mathbb{F}_q$ and let $P=\begin{pmatrix}0&1\\1&0\end{pmatrix}$. Then, by Thas and Payne [19], the set $K = \{(\infty)\} \cup \{(\alpha, c, \alpha HP) || \alpha \in \mathbb{F}_q^2, c \in \mathbb{F}_q\}$ is an ovoid of the generalized quadrangle $\mathcal{S}(\mathcal{C})$.

The existence of these ovoids leads to the following result.

Theorem 3 *A flock generalized quadrangle $GQ(q^2, q)$, with q even, contains minimal blocking sets of excess r, $0 \le r \le q^2 - 1$.*

Proof. Let K be defined as above. Let $p = (\alpha', c', \alpha' HP)$. We determine $\{(\infty), p\}^\perp$. The elements of this set are of type $A^*(t)g$ for $t \in \mathbb{F}_q \cup \{\infty\}$ since they are incident with (∞). They must contain the coset $A(t)p$. Hence, these are the sets $A^*(\infty)p = \{(\alpha', \gamma, \delta) || \gamma \in \mathbb{F}_q, \delta \in \mathbb{F}_q^2\}$ and $A^*(t)p = \{(\alpha + \alpha', c, \alpha K_t + \alpha' HP) || \alpha \in \mathbb{F}_q^2, c \in \mathbb{F}_q\}$.

Now we determine $\{(\infty), p\}^{\perp\perp}$. The elements of this set consist of (∞) and of q elements $g \in \mathcal{K}$ for which we know that $A(t)g \subset A^*(t)p$, $t \in \mathbb{F}_q \cup \{\infty\}$.

From the description of $A^*(\infty)p$ above, necessarily $g = (\alpha', c, \delta)$, for some $c \in \mathbb{F}_q$ and some $\delta \in \mathbb{F}_q^2$. From $A(t)(\alpha', c, \delta) = \{(\alpha + \alpha', \alpha A_t \alpha^T + c + \alpha K_t \alpha'^T, \alpha K_t + \delta) || \alpha \in F_q^2\}, \subset A^*(t)p$, necessarily $\alpha' HP = \delta$. Hence $g = (\alpha', c, \alpha' HP)$ for $c \in \mathbb{F}_q$, and these points belong to the ovoid.

Partitioning the points p of $K \setminus \{(\infty)\}$ into the q^2 sets $\{(\infty), p\}^{\perp\perp}$, we obtain minimal blocking sets of excess $i - 1$ when we replace in K exactly i of these sets by their perp $\{(\infty), p\}^\perp$. ∎

5.2.2 The case q is odd

Consider a flock GQ $\mathcal{S}$ described by using the Knarr construction. Let π be a plane in p^ζ through p, but with $\pi \cap \mathrm{PG}(3,q)$ skew to all the lines of the BLT-set V. Further, let $\overline{\mathrm{PG}(3,q)}$ be a 3-dimensional subspace of PG$(5,q)$ with $\pi \subset \overline{\mathrm{PG}(3,q)} \not\subset p^\zeta$.

Then $K = (\overline{\mathrm{PG}(3,q)} \setminus \pi) \cup \{p\}$ is an ovoid of the GQ $\mathcal{S}$ [19].

Theorem 4 *A flock $GQ(q^2, q)$, with q odd, contains minimal blocking sets of excess r, $0 \le r \le q^2 - 1$.*

Proof. Consider an element $p_1 \in K \setminus \{p\}$. Then $\{p, p_1\}^{\perp\perp}$ is the line pp_1 of PG$(5,q)$ [21, Theorem 7.1] and so, as for q even, $\{p, p_1\}^{\perp\perp}$ is contained in K. Now proceed as in the even case. ∎

Remark

It is possible to give a result similar to Theorem 1 by means of the classical example of a flock GQ; the Hermitian variety $H(3, q^2)$.

If π is a secant plane of $H(3,q^2)$, then the intersection of π with $H(3,q^2)$ is an ovoid of $H(3,q^2)$ consisting of a classical unital $H(2,q^2)$ in π. In the terminology of generalized quadrangles, all points of $H(3,q^2)$ are regular, and the span of any two points of $H(2,q^2)$ is contained in $H(2,q^2)$. In the terminology of projective geometry, take a Baer subline l contained in $H(2,q^2)$ and let L be the line of PG$(3,q^2)$ containing this Baer subline. Applying the polarity τ of $H(3,q^2)$, L is mapped onto a line L^τ intersecting $H(3,q^2)$ in a Baer subline, which is denoted, in the terminology of generalized quadrangles, by $l^\perp$. Again applying τ, L^{τ^2} is mapped onto L and $l^{\perp\perp} = l$.

Let $l_1, \dots, l_m$ be a collection of m Baer sublines contained in $H(2,q^2)$ and assume that $l_i \not\subseteq \cup_{j \neq i} l_j$ for all $i = 1, \dots, m$. Assume $|\cup_{i=1}^m l_i| = m(q+1) - r$.

Then $(H(2,q^2) \setminus \cup_{i=1}^m l_i) \cup_{i=1}^m l_i^\perp$ is a minimal blocking set of excess r.

Let p_0 be a fixed point of $H(2,q^2)$. The remaining q^3 points of $H(2,q^2)$ can be partitioned into q^2 Baer sublines $l_i = \{p_0, p_{i1}, \dots, p_{iq}\}$, $1 \le i \le q^2$.

Using these Baer sublines, it is possible to construct minimal blocking sets $\mathcal{M}_i$, $1 \le i \le q^2$, for which $\mathcal{M}_i$ has excess $i-1$. Each line on p_0 is covered i times, but the lines not passing through p_0 are each covered exactly once.

The variation on the minimal covers of excess three can also be obtained. Let p_1, p_2, p_3 be three non-collinear points of $H(2,q^2)$. Replace the points of the Baer subline $\langle p_1, p_2 \rangle \cap H(2,q^2)$ with those of $\{p_1,p_2\}^\perp$, those of $\langle p_2, p_3 \rangle \cap H(2,q^2)$ with those of $\{p_2,p_3\}^\perp$, and those of $\langle p_1, p_3 \rangle \cap H(2,q^2)$ with those of $\{p_1,p_3\}^\perp$. The resulting minimal blocking set has excess three with three points p_1, p_2, p_3 such that the lines through one of these points all have two points of the blocking set.

5.3 Upper bound on the size of a minimal blocking set

Theorem 5 *Let K be a minimal blocking set in $GQ(s,t)$, $s \ge 2$. Then*

$$|K| \le 1 + st + \frac{st(1+st)}{1+s+t}.$$

Proof. Let $k = |K|$ and suppose that there are θ tangent lines to K. Since K is minimal, necessarily $\theta \ge k$.

Let the secants be $\{L_1, \dots, L_{(1+t)(1+st)-\theta}\}$, and let $k_i = |K \cap L_i|$. Then $\sum_i k_i = |K|(1+t) - \theta$.

Let $r \in K$. Since K has a tangent line at r, we have $|(r^\perp \cap K) \setminus \{r\}| \le st$. So $\sum_i k_i(k_i - 1) \le |K|st$, and so $\sum_i k_i^2 \le |K|(st+t+1) - \theta$.

Now we have $((1+t)(1+st) - \theta) \sum_i k_i^2 - (\sum_i k_i)^2 \ge 0$ which implies

$$\begin{aligned} &|K|(1+t)(1+st)(st+t+1) \\ &\quad \ge \ |K|^2(t+1)^2 + (1+t)(st+1)\theta + \theta|K|(st+t+1) - 2\theta|K|(t+1). \end{aligned}$$

Replacing θ by $|K|$ gives the upper bound on $|K|$. ∎

Remark

For the known cases for the parameters (s,t):
(1) For $\mathrm{GQ}(s,s^2)$, $|K| < s^4 + 2s$ while $|\mathrm{GQ}(s,s^2)| = s^4 + s^3 + s + 1$;
(2) For $\mathrm{GQ}(t^2,t)$, $|K| < t^4 + 2t$ while $|\mathrm{GQ}(t^2,t)| = t^5 + t^3 + t^2 + 1$;
(3) For $\mathrm{GQ}(s,s)$, $|K| < s^3/2 + 3s^2/4 + 5s/8 + 1$ while $|\mathrm{GQ}(s,s)| = s^3 + s^2 + s + 1$;
(4) For $\mathrm{GQ}(q+1,q-1)$, $|K| < q^3/2 + 3q^2/4 - 3q/8 + 1/4$ while $|\mathrm{GQ}(q+1,q-1)| = q^3 + 2q^2$;
(5) For $\mathrm{GQ}(q-1,q+1)$, $|K| < q^3/2 + 3q^2/4 - 3q/8 + 1/4$ while $|\mathrm{GQ}(q-1,q+1)| = q^3$.

6 Small minimal covers in generalized quadrangles

We will now investigate the properties of small minimal covers in generalized quadrangles. We will use the most basic elements of the theory of tight sets (See [13, 14, 15]).

Definition 2 *A subset A of the point set P of $\mathcal{S}$ is called i-*tight *provided $|A| = i(s+1)$ and for each point $x \in A$ it holds that $|x^\perp \cap A| = s + i$. This condition is equivalent to having $|A| = i(s+1)$ and $|x^\perp \cap A| = i$ for each point $x \in P \setminus A$.*

The theory of i-tight sets starts with [16, 1.10.1]. The additional facts about tight sets that we use are:

Theorem 6 *(1) The point set P of $GQ(s,t)$ is (st+1)-tight and the empty set is 0-tight.*
(2) If A and B are disjoint tight sets, then $A \cup B$ is tight.
(3) If A and B are tight with $B \subseteq A$, then $A \setminus B$ is tight.

It is easy to prove that the unique type of 1-tight set is the set of points on a line. Hence it follows that the union of k disjoint lines is k-tight, and the set of points not on k disjoint lines is $(1+st-k)$-tight. It can also be shown that an irreducible 2-tight set (a tight set not the union of two proper subsets each of which is tight) is of the form $X \cup Y$, where $|X| = 1+s$, $|Y| = 1+s$, $Y \subseteq X^\perp$ and $X \subseteq Y^\perp$. Such irreducible 2-tight sets arise when $\{x, y\}$ is a regular pair and $s = t$. Here we may put $X = \{x, y\}^{\perp\perp}$ and $Y = \{x, y\}^\perp$. Such 2-tight sets also arise when $t = s^2$ and $\{x, y, z\}$ is a 3-regular triple of points. In this case we may put $X = \{x, y, z\}^{\perp\perp}$ and $Y = \{x, y, z\}^\perp$.

From now on, let $\mathcal{M}$ be a minimal cover with excess r, $r \leq 2s$. Let $A_i = \{x \in P || x$ is incident with exactly i lines of $\mathcal{M}\}$, $1 \leq i \leq t+1$. Put $a_i = |A_i|$. Then counting first the number of points in P, and then the ordered pairs (x, L), where x is a point incident with the line L of $\mathcal{M}$, we obtain

$$\sum_{j=1}^{t+1} a_j = (1+s)(1+st) \tag{1}$$

$$\sum_{j=1}^{t+1} j a_j = (1+s)(st+1+r) \tag{2}$$

$$\sum_{j=2}^{t+1} (j-1) a_j = r(s+1). \tag{3}$$

Definition 3 *A point of A_1 will be called a* simple *point. A point of A_j, $2 \leq j \leq t+1$, will be called a* multiple point with multiplicity *j.*

From this last equation it follows that there are at most $(1+s)r$ multiple points:

$$|A_2 \cup \cdots \cup A_{t+1}| \leq (1+s)r. \tag{4}$$

Partition $\mathcal{M}$ into two parts $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2$ where $\mathcal{M}_1 = \{L \in \mathcal{M} || L^\perp \cap \mathcal{M} = \{L\}\}$, and where $\mathcal{M}_2 = \{L \in \mathcal{M} || (L^\perp \cap \mathcal{M}) \setminus \{L\} \neq \emptyset\}$.

Let $k = |\mathcal{M}_2|$, so $|\mathcal{M}_1| = (st+1)+r-k$. And put $S_1 = \{x \in P || x$ is on some line $L \in \mathcal{M}_1\}$; $S_2 = P \setminus S_1$.

It follows that S_1 is an $(st + 1 - (k - r))$-tight set, so that S_2 is $(k - r)$-tight, and $k \geq r$. The important principle here is that we now know that each point x of S_2 satisfies $|x^\perp \cap S_2| = s + k - r$.

The next lemma says that no multiple point has multiplicity greater than $r + 1$.

Lemma 2 *The sets A_j are empty when $j \geq r + 2$.*

Proof. By hypothesis $r \leq 2s$ and $j \geq r + 2$. Suppose $z \in A_j$. Let $L_1, \ldots, L_j$ be the lines of $\mathcal{M}_2$ on z, and let x be any point different from z on one of $L_1, \ldots, L_j$, say x is on L_1. Then $x^\perp \cap S_2$ contains the $s + 1$ points of L_1. But $|x^\perp \cap S_2| = s + k - r = s + 1 + (k - r - 1)$. As $x^\perp \cap S_2$ must have $k - r - 1 > k - j$ (because $j > r + 1$) points not on $L_1, \ldots, L_j$, some two of those $k - r - 1$ points must be on the same line of $\mathcal{M}_2$. Since there are no triangles, if x is collinear with two points of S_2 that are on the same line, then all three points are on the same line. Hence x must lie on a line of $\mathcal{M}_2$ in addition to L_1. This forces $z^\perp$ to contain at least $1 + js$ multiple points. Using (4) we have $1 + (r + 2)s \leq 1 + js \leq (1 + s)r$, implying $1 + 2s \leq r$, contradicting our hypothesis. Hence there must be no point $z \in A_j$. ■

Lemma 3 *Suppose $z_1 \in A_i$, $z_2 \in A_j$, $2 \leq i, j$, and $i + j > r + 2$. If $z_1 \sim z_2$ and $x \in z_1z_2 \setminus \{z_1, z_2\}$, then $x \in S_2$.*

Proof. Suppose $x \in z_1z_2 \cap S_1$, where $z_1 \in A_i$, $z_2 \in A_j$, $2 \leq i, j$, and $i + j > r + 2$. Since $x \in S_1$, clearly $z_1z_2 \notin \mathcal{M}$. And $x \notin S_2$ implies that $|x^\perp \cap S_2| = k - r$. But $x^\perp \cap S_2$ must have $k - r - 2$ points covered by $k - (i + j)$ lines of $\mathcal{M}_2$, where $k - r - 2 > k - (i + j)$. Hence some two points of $(x^\perp \cap S_2) \setminus \{z_1, z_2\}$ must lie on the same line L of $\mathcal{M}_2$, forcing x to be on L, that is, $x \in S_2$. ■

Lemma 4 *Suppose $z_1 \in A_i$, $z_2 \in A_j$, $2 \leq i, j$, and $i + j > r + 2$. If $z_1 \sim z_2$ and $x \in z_1z_2 \setminus \{z_1, z_2\}$, then x is a multiple point.*

Proof. Assume the hypothesis. By the preceding lemma each point of z_1z_2 is in S_2. Suppose that $x \in z_1z_2 \setminus \{z_1, z_2\}$ and $x \in S_2 \cap A_1$. So x is on a unique line of $\mathcal{M}_2$ and $|x^\perp \cap S_2| = s + k - r$. First of all, suppose $z_1z_2 \in \mathcal{M}_2$. Let $L_1, \ldots, L_{i-1}$ be the lines of $\mathcal{M}_2$ different from z_1z_2 through z_1, and let $M_1, \ldots, M_{j-1}$ be the lines of $\mathcal{M}_2$ different from z_1z_2 through z_2. By hypothesis z_1z_2 is the unique line of $\mathcal{M}_2$ through x. But $x^\perp \cap S_2$ must contain $(s + k - r) - (s + 1) = k - r - 1$ points of S_2 covered by $k - (i - 1 + j - 1 + 1) = k - i - j + 1$ lines of $\mathcal{M}_2$ different from $L_1, \ldots, L_{i-1}, M_1, \ldots, M_{j-1}, z_1z_2$. Since $k - r - 1 > k - i - j + 1$, x must lie on a second line of $\mathcal{M}_2$.

Now suppose $z_1z_2 \notin \mathcal{M}_2$. We know each point x of z_1z_2 is on at least one line of $\mathcal{M}_2$. Also, $x^\perp \cap S_2$ contains the $1 + s$ points of z_1z_2 and s additional points on a line L of $\mathcal{M}_2$ through x. Since each point of $z_1z_2 \setminus \{z_1, z_2\}$ is on at least one line of $\mathcal{M}_2$, there are at most $k - (i + j) - (s - 1)$ lines of $\mathcal{M}_2$, all different from L, which can contain the remaining $s + k - r - (1 + 2s) = k - r - s - 1$ points of $x^\perp \cap S_2$. But again $k - r - s - 1 > k - i - j - s + 1$, so at least two of those $k - r - s - 1$ points of $x^\perp \cap S_2$ must belong to the same line of $\mathcal{M}_2$. Note that this line is different from L. Hence, x belongs to a second line of $\mathcal{M}_2$. The point x is a multiple point. ■

Remark

In the preceding lemma it must be the case that $z_1z_2 \notin \mathcal{M}_2$, since otherwise because each point of z_1z_2 is a multiple point, z_1z_2 could be removed from $\mathcal{M}_2$ and $\mathcal{M}\backslash\{z_1z_2\}$ would still be a cover. This proves the following corollary.

Corollary

If $z \in A_{r+1}, 1 \leq r \leq 2s$, and $z \in L \in \mathcal{M}_2$, then the only multiple point on L is z.

Theorem 7 *Let $\mathcal{M}$ be a minimal cover with excess r, $1 \leq r \leq 2s$.*
(1) If $z \in A_{r+1}$, then $z^\perp$ contains all multiple points; on a line passing through z, either z is the only multiple point, or all points are multiple.
(2) If $|A_{r+1}| = a_{r+1} \geq 2$, then there is a line $L \notin \mathcal{M}$ such that the set of points incident with L is the set of multiple points, and they all have multiplicity $r+1$.

Proof. Suppose $z \in A_{r+1}$ and let $L_0, L_1, \ldots, L_r$ be the lines of $\mathcal{M}_2$ through z. Suppose there is a multiple point w, $w \not\sim z$. Since L_r contains no multiple point other than z, the line wx through w meeting L_r at some point x must not be in $\mathcal{M}_2$. Clearly $x^\perp \cap S_2$ contains the $s+1$ points of $zx = L_r$, the multiple point w, and $s+k-r-(s+1)-1 = k-r-2$ points on at most $k-(r+1+2) = k-r-3$ lines of $\mathcal{M}_2$, different from $L_0, \ldots, L_r$ and not passing through x. This forces x to be collinear with two points of a line of $\mathcal{M}_2$. Since this line is different from L_r, x belongs to at least two lines of $\mathcal{M}_2$, and so x is a multiple point.

This impossibility shows that each multiple point w must be collinear with z. The remaining part follows from Lemma 4.

Now suppose there are two points $z_1, z_2 \in A_{r+1}$. So $z_1 \sim z_2$, and each point of z_1z_2 is a multiple point. Let z_1 belong to the lines $L_0, \ldots, L_r$ of $\mathcal{M}$. The line $z_1^\perp \cap z_2^\perp = z_1z_2$ must contain all multiple points. Hence $a_2+a_3+\cdots+a_{r+1} = s+1$, so that by (1) $a_1 = (1+s)st$. Let $L = z_1z_2 = \{z_1, z_2, \ldots, z_{s+1}\}$ be the line of multiple points. Since $L \notin \mathcal{M}_2$, and since each line of $\mathcal{M}_2$ is incident with a unique point of L, the points of $z_1^\perp \cap S_2$ must be exactly the points of $L_0, L_1, \ldots, L_r, L$. Hence $|z_1^\perp \cap S_2| = 1+s(r+2) = s+k-r$ implies that $k = 1+rs+r+s = (r+1)(s+1)$. Since no point of L is on more than $r+1$ lines of $\mathcal{M}_2$, it must be that each point of L is in A_{r+1}. ∎

Corollary

Let $\mathcal{M}$ be a minimal cover with excess 1. Then there exists a line L in the generalized quadrangle, not belonging to the minimal cover, such that all the points of L have multiplicity two.

Proof. Specializing the preceding results, we see that $a_j = 0$ for $j \geq 3$. So by (3) we have $a_2 = 1+s$. Moreover, by Theorem 7, there is a line $L \notin \mathcal{M}$ such that A_2 is the set of points incident with L. So each point of L is on two lines of $\mathcal{M}_2$, and each point of $\mathcal{S}$ not covered by these $2(1+s)$ lines of $\mathcal{M}_2$ is on a unique line of $\mathcal{M}_1$. ∎

Remark

The preceding theorem now makes it possible to give an example of a generalized quadrangle with spreads, but with no minimal cover of excess 1.

The unique GQ(5,3) has 24 spreads and 4608 ovoids [14], but we will show that it has no minimal cover with excess 1.

For a minimal cover $\mathcal{M}$ with excess 1 must be of the following type. There is some line K_0 with multiple points $x_0, x_1, \ldots, x_5$. Then the other lines through x_i can be labeled so that x_i is on K_0, L_i, M_i, N_i, $0 \leq i \leq 5$, and $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2$ where $\mathcal{M}_2 = \{L_0, M_0, L_1, M_1, \ldots, L_5, M_5\}$. So $k = 12$, $k - r = 11$, and $st + 1 - (k - r) = 5$. So there must be 5 lines $K_1, \ldots, K_5$ in $\mathcal{M}_1$ that form a partial spread and cover the points of $N_0, \ldots, N_5$ not on K_0. Hence $\{K_0, K_1, \ldots, K_5\}^\perp = \{N_0, \ldots, N_5\}$.

But then by (the point-line dual of) [16, 1.3.6(i)], it must be that $3 \geq 5$, a patent impossibility. So GQ(5,3) has no minimal cover with excess 1.

7 Minimal covers with excess 2

To conclude the study of small minimal covers in generalized quadrangles, we look more in detail to the minimal covers of excess two.

Let $\mathcal{M}$ be a minimal cover with excess 2, so $r = 2$. Specializing the results of Section 6, we see that $a_j = 0$ for $j \geq 4$. And by (3) we have $a_2 + 2a_3 = 2(1 + s)$.

Since $3 + 2 = 5 > r + 2$, each point on a line joining a triple point with a multiple point is a multiple point. And each triple point is collinear with each multiple point. Moreover, as all triple points are collinear, they must all lie on one line.

There are a number of cases which have to be treated separately.

Case 1. $a_3 = s + 1$. In this case there are no double points, and there is a line L all of whose points are triple points. Clearly $L \notin \mathcal{M}_2$, since otherwise $\mathcal{M}$ would not be minimal.

Any GQ$(s, 3)$ has minimal covers $\mathcal{M}$ for which $a_3 = s + 1$. Namely, for a fixed line L of GQ$(s, 3)$, $\mathcal{M} = L^\perp \setminus \{L\}$ is a minimal cover for which $a_3 = s + 1$. For a GQ$(s, 3)$, when $a_3 = s + 1$, also $\mathcal{M}_2 = \mathcal{M}$. Finally, also the minimal covers with $r = 2$ described in Theorems 3 and 4 are of this type.

Case 2. $1 \leq a_3 \leq s$. Then $a_3 = 1$ since $a_3 \geq 2$ implies $a_3 = s + 1$ (Theorem 7).

So with $a_3 = 1$, it follows that $a_2 = 2s$, and the one point $z \in A_3$ is collinear with all $2s$ points of A_2, and any line incident with z and one point of A_2 must have s points of A_2. Such a line cannot be in $\mathcal{M}$, for otherwise $\mathcal{M}$ would not be minimal. So we have z on three lines M_1, M_2, M_3 of $\mathcal{M}$ that have no further multiple points. And z is on two lines L_1, L_2 not in $\mathcal{M}$, each of whose s other points are on two lines of $\mathcal{M}$. So there are $k = 3 + 4s$ lines in $\mathcal{M}_2$, and S_2 is $(1 + 4s)$-tight. Also S_1 is $s(t - 4)$-tight and is covered by $s(t - 4)$ pairwise non-concurrent lines, none of which meets any of the lines in $\mathcal{M}_2$.

We have an example of this type, with $\mathcal{M} = \mathcal{M}_2$. In GQ(2,4), the unique triple point will be $z = (12)$. The three lines through z in $\mathcal{M}_2$ are the three lines in the syntheme-duad subquadrangle GQ(2,2). One other line through z has the points 1 and $2'$; the other has points $1'$ and 2. So put $\mathcal{M}_1 = \emptyset$ and $\mathcal{M}_2 = \{(12, 34, 56), (12, 35, 46), (12, 36, 45), (1, 13, 3'), (1, 14, 4'), (2', 23, 3), (2', 24, 4), (1', 15, 5), (1', 16, 6), (2, 25, 5'), (2, 26, 6')\}$.

Case 3. $a_3 = 0; a_2 = 2(1 + s)$**.**

The general theory does not yet determine what form this type of minimal cover must assume. However, we give one that occurs in GQ(2,2).

Put $\mathcal{M}_1 = \{(16, 23, 45)\}$; $\mathcal{M}_2 = \{(12, 34, 56), (34, 15, 26), (15, 36, 24), (36, 25, 14), (25, 46, 13), (46, 12, 35)\}$.

The first two points in each line of $\mathcal{M}_2$ are points of A_2 and the third is a point of A_1. Note that the points of A_2 are the corners of a hexagon.

References

[1] L. Bader and G. Lunardon. On the flocks of $\mathcal{Q}^+(3, q)$. *Geom. Dedicata* 29 (1989), 177-183.

[2] L. Bader, G. Lunardon, and J. A. Thas. Derivation of flocks of quadratic cones. *Forum Math.* 2 (1990), 163-174.

[3] A. Blokhuis. On the size of a blocking set in PG$(2, q)$. *Combinatorica* 14 (1994), 273-276.

[4] A. Blokhuis. Blocking sets in Desarguesian planes. In *Paul Erdős is Eighty*, Volume 2 (D. Miklós, V.T. Sós and T. Szőnyi, (eds.)), Bolyai Soc. Math. Studies, 2 (1996), 133-155.

[5] A. Blokhuis and K. Metsch. On the size of a maximal partial spread. *Des. Codes Cryptogr.* 3 (1993), 187-191.

[6] A. A. Bruen and J. A. Thas. Blocking sets. *Geom. Dedicata* 6 (1977), 193-203.

[7] J. Eisfeld. Private communication (1997).

[8] H. Gevaert, N. L. Johnson, and J. A. Thas. Spreads covered by reguli. *Simon Stevin* 62 (1988), 51-62.

[9] J. W. P. Hirschfeld. *Projective Geometries over Finite Fields*, Clarendon Press, Oxford 1979.

[10] W. M. Kantor. Ovoids and translation planes. *Canad. J. Math.* 34 (1982), 1195-1203.

[11] N. Knarr. A geometric construction of generalized quadrangles of polar spaces of rank three. *Results Math.* 21 (1992), 332-344.

[12] K. Metsch. Improvement of Bruck's completion theorem. *Des. Codes and Cryptogr.* 1 (1991), 99-116.

[13] S. E. Payne. Tight point sets in finite generalized quadrangles. *Congr. Numer.* 60 (1987), 243-260.

[14] S. E. Payne. The generalized quadrangle with $(s, t) = (3, 5)$. *Congr. Numer.* 77 (1990), 5-29.

[15] S. E. Payne. Tight point sets in finite generalized quadrangles II. *Congr. Numer.* 77 (1990), 31-41.

[16] S. E. Payne and J. A. Thas. *Finite generalized quadrangles.* Pitman Pub. Co., London, 1984.

[17] J. A. Thas. Flocks of non-singular ruled quadrics in PG$(3, q)$. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* (8) 59 (1975), 83-85.

[18] J. A. Thas. Flocks, maximal exterior sets and inversive planes. In *Finite Geometries and Combinatorial Designs*, *Contemp. Math.* 111 (1990), 187-218.

[19] J. A. Thas and S. E. Payne. Spreads and ovoids in finite generalized quadrangles, *Geom. Dedicata* 52 (1994), 227-253.

[20] J. A. Thas. Projective geometry over a finite field. In *Handbook of Incidence Geometry, Buildings and Foundations*, (F. Buekenhout, (ed.)), Chapter 7, pp. 295-348. Amsterdam, North-Holland, 1997.

[21] J. A. Thas and H. Van Maldeghem. Generalized quadrangles and the axiom of Veblen. In *Geometry, Combinatorial Designs and Related Structures.* Proceedings of the First Pythagorean Conference (Spetses, Greece, June 1-7, 1996). London Mathematical Society Lecture Note Series 245, Cambridge University Press 1997, 241-253.

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