Arcs and ovals in infinite K-clan geometry

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Abstract

For a finite field $\mathrm{GF}(q)$, to a q-clan of matrices there are associated generalized quadrangles, flocks of quadratic cones in $\mathrm{PG}(3, q)$, translation planes and, for q even, ovals in $\mathrm{PG}(2, q)$. The connections with generalized quadrangles, flocks and translation planes have recently been extended to the case of an infinite field K, under certain extra assumptions. In this note we extend the theory of ovals in $\mathrm{PG}(2, q)$ associated with q-clans, q even, to ovals in $\mathrm{PG}(2, K)$ associated with K-clans for (infinite) fields K of characteristic 2. Again, certain extra assumptions on the field K are made.

1 Introduction

The term *q-clan geometry* is often used to refer to the well-developed theory of flocks of quadratic cones and their associated generalized quadrangles and translation planes, over a finite field $\mathrm{GF}(q)$. In [4] F. De Clerck and H. Van Maldeghem used the coordinatization of a generalized quadrangle to extend this theory to the case of an infinite field K. They introduced a natural definition of a K-clan as a family of matrices and showed that there is an associated generalized quadrangle if and only if the associated flock is derivable.

Moving to the point of view of coset geometries and following the work of W.M. Kantor and S.E. Payne for finite fields, L. Bader and S.E. Payne [1] have further

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investigated K-clan geometry. They first defined (possibly infinite) 4-gonal families, which are equivalent to elation generalized quadrangles, and called a K-clan *4-gonal* if it gives rise to a generalized quadrangle.

S.E. Payne [9] and W. Cherowitzo, T. Penttila, I. Pinneri and G.F. Royle [3] have constructed the so-called herds of ovals, associated with q-clans when q is even (see also the work of L. Storme and J.A. Thas [12]). Our aim in this study is to extend the theory of herds to infinite fields (of characteristic 2). However, as occurred in the analogous extension of the theory of flocks, we need to add some extra hypotheses on the field in order to obtain ovals.

In Section 2 we present definitions and results, most of which appear in [1, 4], which are needed in the rest of the paper. As we are only interested in fields K of even characteristic, we rephrase known results for our particular purposes.

Throughout Section 3 we assume that K is a full field of characteristic 2. Using arguments similar to those found in [3], we show that to a partial K-clan $\mathcal{C}$ there corresponds a family of arcs in $\mathrm{PG}(2, K)$. If $\mathcal{C}$ is a K-clan then the arcs are complete, and if $\mathcal{C}$ is a 4-gonal K-clan then the arcs are ovals.

In Section 4 we suppose the field K is perfect, and we construct a family of ovals in $\mathrm{PG}(2, K)$ associated with a 4-gonal K-clan.

Section 5 deals with subquadrangles. As in the finite case, a generalized quadrangle associated with a 4-gonal K-clan admits a family of subquadrangles. If the field is perfect, then each of these is isomorphic to the infinite analogue of Tits' generalized quadrangle, usually denoted by $T_2(\mathcal{O})$, for an oval $\mathcal{O}$ in $\mathrm{PG}(2, K)$.

Finally, in Section 6 we briefly recall the known classes of examples of 4-gonal K-clans for K an infinite field of characteristic 2 and display their associated ovals.

2 Preliminaries

We recall the construction of a group coset geometry, first suggested by Kantor [8] in the finite case. Let s, t be cardinal numbers. Let $\mathcal{G}$ be a group, let $\mathcal{I}$ be a set of indices of order $t + 1$ (that is, there is a bijection from $\mathcal{I}$ to $X \cup \{*\}$ where X is any set of order t and $*$ is a symbol not in X) and suppose there exist two families $\mathcal{F} = \{A(i) : i \in \mathcal{I}\}$ and $\mathcal{F}^* = \{A^*(i) : i \in \mathcal{I}\}$ of subgroups of $\mathcal{G}$ with $A(i) < A^*(i)$, $|A(i)| = |A(j)| = s$, $|A^*(i)| = |A^*(j)|$, $[A^*(i) \colon A(i)] = [A^*(j) \colon A(j)] = t$ for all $i, j \in \mathcal{I}$ and satisfying:
(K1) $A(i)A(j) \cap A(k) = 1$;
(K2) $A^*(i) \cap A(j) = 1$;
(K3) $A^*(i)A(j) = \mathcal{G}$;
(K4) $A^*(i) = A(i) \cup \{A(i)g : g \in \mathcal{G} \text{ and } A(i)g \cap A(j) = \emptyset \text{ for all } j \neq i\}$
$= A(i) \cup \{g \in \mathcal{G} : A(i)g \cap A(j) = \emptyset \text{ for all } j \neq i\}$
for all distinct $i, j, k \in \mathcal{I}$.
We remark that if $\mathcal{G}$ is finite and (K1) and (K2) hold then (K3) and (K4) are trivially satisfied.

The triple $(\mathcal{G}, \mathcal{F}, \mathcal{F}^*)$ is called a *Kantor family,* and has a corresponding group coset geometry $\mathcal{Q}(\mathcal{G}, \mathcal{F}, \mathcal{F}^*)$ defined as follows: *Points* are: (i) Elements $g \in \mathcal{G}$; (ii) Cosets $A^*(i)g$ for $i \in \mathcal{I}$ and $g \in \mathcal{G}$; (iii) The symbol (∞). *Lines* are: (a) Cosets $A(i)g$ for $i \in \mathcal{I}$ and $g \in \mathcal{G}$; (b) Symbols $[A(i)]$ for $i \in \mathcal{I}$. *Incidence* is: the point (∞)

is on each line $[A(i)]$, the point $A^*(i)g$ is on the line $[A(i)]$ and each line $A(i)h$ for which $A(i)h \subseteq A^*(i)g$ and the point g is on each line $A(i)g$. There are no further incidences.

Theorem 1 ([8, 1]) *Let $(\mathcal{G}, \mathcal{F}, \mathcal{F}^*)$ be a Kantor family. Then the corresponding group coset geometry $\mathcal{Q}(\mathcal{G}, \mathcal{F}, \mathcal{F}^*)$ is an elation generalized quadrangle with parameters (s,t), base point (∞) and elation group $\mathcal{G}$. Conversely, any elation generalized quadrangle with parameters (s,t), base point (∞) and elation group $\mathcal{G}$ is isomorphic to the group coset geometry of a suitable Kantor family comprising subgroups of $\mathcal{G}$.* ■

For the next example, we need the following definitions. Let K be a field. An *arc* in the projective plane $\mathrm{PG}(2,K)$ is a set of points, no three collinear, while a *complete arc* is an arc which is not properly contained in another arc. A line is an *external, tangent* or *secant* of an arc according as it meets the arc in 0, 1 or 2 points. An *oval* is an arc which admits a unique tangent at each of its points.

Example 1 Let K be a field and let $\mathcal{O} = \{(1, b_t, c_t) : \ t \in K\} \cup \{(0,0,1)\}$ be an oval in $\mathrm{PG}(2,K)$, with tangents concurrent in the point $N = (0,1,0)$. (In particular, $\mathcal{O} \cup \{N\}$ is an arc.) Let G denote the group $K \times K \times K$ under componentwise addition and define the subgroups:

$$\begin{aligned} A(\infty) &= \{(0,0,z) : \ z \in K\} \\ A(t) &= \{(x, xb_t, xc_t) : \ x \in K\} \text{ for } t \in K \\ Z &= \{(0,y,0) : \ y \in K\} \\ A^*(t) &= A(t)Z \quad \text{ for } t \in K \cup \{\infty\}. \end{aligned}$$

If we write $\mathcal{F} = \{A(t) : \ t \in K \cup \{\infty\}\}$ and $\mathcal{F}^* = \{A^*(t) : \ t \in K \cup \{\infty\}\}$ then we will now show that $(G, \mathcal{F}, \mathcal{F}^*)$ is a Kantor family, and we denote the corresponding generalized quadrangle by $T_2(\mathcal{O})$, following Tits (see [5] or [14]). For $t \in K \cup \{\infty\}$ we have: $A(t) < A^*(t)$, $|A(t)| = |K|$, $|A^*(t)| = |K|^2$ and $[A^*(t) : A(t)] = |K|$.

We use the map $\phi : \ G \to \mathrm{PG}(2,K)$, $(x,y,z) \mapsto (x,y,z)$. Under this map, $\phi : \ A(\infty) \mapsto (0,0,1) = P_\infty$ and $A(t) \mapsto (1, b_t, c_t) = P_t$ for $t \in K$; so $\phi(\mathcal{F}) = \mathcal{O}$. Further, $\phi : \ Z \mapsto (0,1,0) = N$ and for $t \in K \cup \{\infty\}$ we have $\phi : \ A^*(t) \mapsto P_t N$.

Since $\mathcal{O}$ is an arc, no three of its points have linearly dependent coordinate vectors, hence (K1) holds. Second, since for each $t \neq u$ we have $P_u \notin P_t N$, so $A(u) \cap A^*(t) = 1$ and (K2) holds. Since for each $t \neq u$ the point P_u and the line $P_t N$ together span $\mathrm{PG}(2,K)$, (K3) follows. Finally, (K4) follows since for each $t \in K \cup \{\infty\}$, the tangent $P_t N$ comprises the point P_t together with every point P for which PP_t is tangent to $\mathcal{O}$. ■

We will need the following lemma which deals with the existence of Kantor families on subgroups of $\mathcal{G}$.

Lemma 1 *Let $\mathcal{G}$ be a group and let $(\mathcal{G}, \mathcal{F}, \mathcal{F}^*)$ be a Kantor family, where $\mathcal{F} = \{A(i) : \ i \in \mathcal{I}\}$ and $\mathcal{F}^* = \{A^*(i) : \ i \in \mathcal{I}\}$ are families of subgroups of $\mathcal{G}$, $\mathcal{I}$ is a set of indices of order $t+1$ and $|A(i)| = s$ for $i \in \mathcal{I}$. Let $\mathcal{H}$ be a subgroup of $\mathcal{G}$, let*

$B(i) = A(i) \cap \mathcal{H}$ and $B^*(i) = A^*(i) \cap \mathcal{H}$ for $i \in \mathcal{I}$ and let $\mathcal{I}' = \{i \in \mathcal{I} : B(i) \neq 1\}$. Suppose there exist cardinal numbers s', t' such that $|\mathcal{I}'| = t' + 1$, $|B(i)| = |B(j)| = s'$, $|B^*(i)| = |B^*(j)|$ and $[B^*(i) \colon B(i)] = [B^*(j) \colon B(j)] = t'$ for all $i, j \in \mathcal{I}'$. Let $\mathcal{F}' = \{B(i) : i \in \mathcal{I}'\}$ and $\mathcal{F}^{*\prime} = \{B^*(i) : i \in \mathcal{I}'\}$.
(1) If $\mathcal{H}$ is finite then $(\mathcal{H}, \mathcal{F}', \mathcal{F}^{*\prime})$ is a Kantor family.
(2) If $B^*(i)B(j) = \mathcal{H}$ for all $i, j \in \mathcal{I}'$, $i \neq j$, then $(\mathcal{H}, \mathcal{F}', \mathcal{F}^{*\prime})$ is a Kantor family.

Proof.
Proof of (K1). Let $x \in B(i)B(j) \cap B(k)$ for distinct $i, j, k \in \mathcal{I}'$. Then $x \in A(i)A(j) \cap A(k)$ for distinct $i, j, k \in \mathcal{I}$, so $x = 1$ since $(\mathcal{G}, \mathcal{F}, \mathcal{F}^*)$ is a Kantor family.
Proof of (K2). Let $x \in B^*(i) \cap B(j)$ for distinct $i, j \in \mathcal{I}'$. Then $x \in A^*(i) \cap A(j)$ for distinct $i, j \in \mathcal{I}$, so $x = 1$ since $(\mathcal{G}, \mathcal{F}, \mathcal{F}^*)$ is a Kantor family.
If $\mathcal{H}$ is finite then (K3) and (K4) are trivially satisfied, hence part (1) holds.

Suppose now that $\mathcal{H}$ is not finite, and suppose that $B^*(i)B(j) = \mathcal{H}$ for all $i, j \in \mathcal{I}'$, $i \neq j$. We have already shown that (K1) and (K2) hold.
Proof of (K3): The additional hypothesis above is precisely (K3).
Proof of (K4): Fix $i \in \mathcal{I}'$. For $h \in B^*(i)$, since $B^*(i) = A^*(i) \cap \mathcal{H}$, we have $h \in A^*(i) = A(i) \cup \{A(i)g : g \in \mathcal{G} \text{ and } A(i)g \cap A(j) = \emptyset \text{ for all } j \neq i\}$. If $h \in A(i)$ then $h \in A(i) \cap \mathcal{H} = B(i)$. Otherwise, there exists $g \in \mathcal{G}$ such that $h \in A(i)g$ where $A(i)g \cap A(j) = \emptyset$ for all $j \in \mathcal{I}$ with $j \neq i$. Now $h \in A(i)g$ implies $A(i)g = A(i)h$, so $h \in A(i)h$ and $A(i)h \cap A(j) = A(i)g \cap A(j) = \emptyset$ for all $j \in \mathcal{I}$ with $j \neq i$ implies that $B(i)h \cap B(j) = \emptyset$ for all $i \in \mathcal{I}'$ with $j \neq i$. We have therefore shown that $B^*(i) \subseteq B(i) \cup \{B(i)g : g \in \mathcal{H} \text{ and } B(i)g \cap B(j) = \emptyset \text{ for all } j \neq i\}$.
For the reverse inclusion, first recall that $B(i) \subseteq B^*(i)$. Choose $h \in \mathcal{H}$ such that $B(i)h \cap B(j) = \emptyset$ for all $j \in \mathcal{I}'$ with $j \neq i$. We will prove that $A(i)h \cap A(j) = \emptyset$ for all $j \in \mathcal{I}'$ with $j \neq i$; hence showing that $h \in A^*(i)$ and therefore $h \in B^*(i)$. Suppose, aiming for a contradiction, that there exists $g \in A(i)h \cap A(j)$ for some $j \in \mathcal{I}'$ with $j \neq i$. Since $B(i)h \cap B(j) = \emptyset$ it follows that $g \notin \mathcal{H}$. Let $g_i \in A(i)$ and $g_j \in A(j)$ be such that $g = g_i h = g_j$. Since $g \notin \mathcal{H}$ and $h \in \mathcal{H}$, we have $g_i, g_j \notin \mathcal{H}$. On the other hand, since $B^*(i)B(j) = \mathcal{H}$, there exist $h_i \in B^*(i) \subseteq A^*(i)$ and $h_j \in B(j) \subseteq A(j)$ such that $h = h_i h_j$. We have $h = h_i h_j = g_i^{-1} g_j$; so $g_i h_i = g_j h_j^{-1} \in A^*(i) \cap A(j) = 1$ and hence $g_j = h_j \in \mathcal{H}$ and $g_i = h_i^{-1} \in \mathcal{H}$; giving the required contradiction. ∎

We now review a construction method for Kantor families using collections of matrices over a field K. These are analogous to q-clans in the case that $K = \mathrm{GF}(q)$ (see, for example, [1, 9, 10, 14]). Although some of the results below can be stated in more generality, we will restrict our attention to fields of characteristic 2.

Let K be a field of characteristic 2 and let $\mathcal{C} = \left\{ A_t = \begin{pmatrix} x_t & y_t \\ 0 & z_t \end{pmatrix} : t \in K \right\}$ be a collection of matrices with $x_t, y_t, z_t \in K$. Let $\mathcal{G} = \{(\alpha, c, \beta) : \alpha, \beta \in K^2, c \in K\}$ and define a multiplication $\cdot$ on $\mathcal{G}$ by

$$(\alpha, c, \beta) \cdot (\alpha', c', \beta') = (\alpha + \alpha', c + c' + \beta P \alpha'^T, \beta + \beta')$$

where $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $(\mathcal{G}, \cdot)$ is a group (in fact it is isomorphic to the group used in [1, Section 3], under the map $(\alpha, c, \beta) \mapsto (\alpha, c, \beta P)$). We define subgroups

$$\begin{array}{rcll} Z & = & \{(0,c,0) : c \in K\} & \\ A(\infty) & = & \{(0,0,\beta) : \beta \in K^2\} & \\ A(t) & = & \{(\alpha, \alpha A_t \alpha^T, \alpha y_t) : \alpha \in K^2\} & \text{for } t \in K \\ A^*(t) & = & A(t)Z \quad \text{for } t \in K \cup \{\infty\}. & \end{array}$$

Let $\mathcal{F} = \{A(t) : t \in K \cup \{\infty\}\}$ and $\mathcal{F}^* = \{A^*(t) : t \in K \cup \{\infty\}\}$.

In this case we denote the group coset geometry $\mathcal{Q}(\mathcal{G}, \mathcal{F}, \mathcal{F}^*)$ by $\mathrm{GQ}(\mathcal{C})$. De Clerck and Van Maldeghem [4] and Bader and Payne [1] have investigated the conditions on $\mathcal{C}$ under which $(\mathcal{G}, \mathcal{F}, \mathcal{F}^*)$ is a Kantor family and therefore $\mathrm{GQ}(\mathcal{C})$ is a generalized quadrangle.

Theorem 2 ([4, 1]) *Let K be a field of characteristic 2 and let*

$$\mathcal{C} = \left\{ A_t = \begin{pmatrix} x_t & y_t \\ 0 & z_t \end{pmatrix} : t \in K \right\}$$

be a collection of matrices with $x_t, y_t, z_t \in K$. Then $\mathrm{GQ}(\mathcal{C})$ is a generalized quadrangle if and only if the following conditions hold for all $u \in K$ and for all $\alpha \in K^2 \setminus \{(0,0)\}$:
(a) The map $K \to K, t \mapsto \alpha A_t \alpha^T$ is injective;
(b) The map $K \to K, t \mapsto \alpha A_t \alpha^T$ is surjective;
(c) The map $K \to K, t \mapsto y_t$ is injective;
(d) The map $K \setminus \{u\} \to K \setminus \{0\}, t \mapsto \frac{\alpha(A_t + A_u)\alpha^T}{(y_t+y_u)^2}$ is bijective. ∎

Similarly, the relationship between flocks of a quadratic cone in $\mathrm{PG}(3,q)$ and q-clans [13] suggests the following generalization to any field K.

Let K be a field. A *flock* of a quadratic cone $\mathcal{Q}$ with vertex V in $\mathrm{PG}(3,K)$ is a collection of planes whose intersections with $\mathcal{Q}$ partition the points of $\mathcal{Q} \setminus V$ into disjoint irreducible conics. A *partial flock* of $\mathcal{Q}$ is a collection of planes whose intersections with $\mathcal{Q}$ are pairwise disjoint irreducible conics. In $\mathrm{PG}(3,K)$, let $\mathcal{Q} = \{(X_0, X_1, X_2, X_3) : X_2^2 = X_0 X_1\}$ be the quadratic cone with vertex $V = (0,0,0,1)$. Given a collection $\mathcal{C} = \left\{ A_t = \begin{pmatrix} x_t & y_t \\ 0 & z_t \end{pmatrix} : t \in K \right\}$ of matrices with $x_t, y_t, z_t \in K$, we can define a collection of planes in $\mathrm{PG}(3,K)$ by $\mathcal{F}(\mathcal{C}) = \{x_t X_0 + z_t X_1 + y_t X_2 + X_3 = 0 : t \in K\}$. De Clerck and Van Maldeghem [4] sought conditions on $\mathcal{C}$ in order that $\mathcal{F}(\mathcal{C})$ should be a flock of $\mathcal{Q}$.

Theorem 3 ([4]) *Let K be a field of characteristic 2 and let*

$$\mathcal{C} = \left\{ A_t = \begin{pmatrix} x_t & y_t \\ 0 & z_t \end{pmatrix} : t \in K \right\}$$

be a collection of matrices with $x_t, y_t, z_t \in K$. Let $\mathcal{F}(\mathcal{C}) = \{x_t X_0 + z_t X_1 + y_t X_2 + X_3 = 0 : t \in K\}$. Then $\mathcal{F}(\mathcal{C})$ is a partial flock of the quadratic cone $\mathcal{Q} : X_2^2 = X_0 X_1$ if and only if condition (a) of Theorem 2 holds. Further, $\mathcal{F}(\mathcal{C})$ is a flock if and only if conditions (a) and (b) of Theorem 2 hold. ∎

De Clerck and Van Maldeghem [4] coined the term K-clan for a collection of upper-triangular matrices over K which correspond in this way to a flock of a quadratic cone in $\mathrm{PG}(3,K)$. Also, Bader and Payne [1, Section 4] argued that there is no loss of generality in assuming that each matrix is upper-triangular and that A_0 is the zero matrix. Thus we have the following definitions. Let K be a field of characteristic 2, and let $\mathcal{C} = \left\{ A_t = \begin{pmatrix} x_t & y_t \\ 0 & z_t \end{pmatrix} : t \in K \right\}$ be a collection of (upper triangular) matrices with $x_t, y_t, z_t \in K$. Then $\mathcal{C}$ is a *partial K-clan* if condition (a) of Theorem 2 holds. Further, $\mathcal{C}$ is a *K-clan* if conditions (a) and (b) of Theorem 2 hold.

In particular, if $\mathcal{C}$ is a partial K-clan then it is straightforward to verify that each of the maps $t \mapsto x_t$ and $t \mapsto z_t$ is injective. If $\mathcal{C}$ is a K-clan then each of the maps $t \mapsto x_t$ and $t \mapsto z_t$ is bijective.

A K-clan $\mathcal{C}$ is called *4-gonal* if the associated group coset geometry $\mathrm{GQ}(\mathcal{C})$ is a generalized quadrangle, that is, if $\mathcal{Q}(\mathcal{G}, \mathcal{F}, \mathcal{F}^*)$ is a Kantor family, which is if and only if $\mathcal{C}$ satisfies properties (a)–(d) of Theorem 2.

3 Arcs from partial K-clans over full fields

Let K be a field of characteristic 2 and let $\mathcal{C}_1 = \{k \in K : x^2+x+k$ is irreducible over K$\}$. We say that K is *full* if for every $k_1, k_2 \in \mathcal{C}_1$ we have $k_1 + k_2 \notin \mathcal{C}_1$. For example, the finite field $\mathrm{GF}(2^h), h \geq 1$, is full and in that case $\mathcal{C}_1$ is the set of elements of (absolute) trace 1. In fact, the union of the finite fields $\mathrm{GF}(2^h)$ for all odd $h \geq 1$, is full.

Throughout this section we assume that K is a full field of characteristic 2. We first give an alternative condition for a K-clan which is useful in this case. It follows immediately by arguments analogous to those of Thas [13] for finite K.

Lemma 2 ([13]) *Let* $\mathcal{C} = \left\{ A_t = \begin{pmatrix} x_t & y_t \\ 0 & z_t \end{pmatrix} : t \in K \right\}$ *be a collection of matrices with $x_t, y_t, z_t \in K$ such that the map $K \to K, t \mapsto y_t$ is injective. Then $\mathcal{C}$ is a partial K-clan if and only if $(x_t + x_u)(z_t + z_u)(y_t + y_u)^{-2} \in \mathcal{C}_1$ for all $t, u \in K$ with $t \neq u$.* ■

Theorem 4 *Let K be a full field of characteristic 2 and let*

$$\mathcal{C} = \left\{ A_t = \begin{pmatrix} x_t & y_t \\ 0 & z_t \end{pmatrix} : t \in K \right\}$$

be a collection of matrices with $x_t, y_t, z_t \in K$. For $\alpha \in K^2 \setminus \{(0,0)\}$ define the following sets of points in $\mathrm{PG}(2,K)$:

$$\begin{aligned} \mathcal{O}_\alpha &= \{(1, y_t^2, \alpha A_t \alpha^T) : t \in K\} \cup \{(0,1,0)\} \\ \mathcal{H}_\alpha &= \mathcal{O}_\alpha \cup \{(0,0,1)\}. \end{aligned}$$

(i) $\mathcal{C}$ is a partial K-clan and $t \mapsto y_t$ is injective if and only if $\mathcal{H}_\alpha$ is an arc for all $\alpha \in K^2 \setminus \{(0,0)\}$.
(ii) If $\mathcal{C}$ is a K-clan and $t \mapsto y_t$ is injective then $\mathcal{H}_\alpha$ is a complete arc for all

$\alpha \in K^2 \setminus \{(0,0)\}$.
(iii) $\mathcal{C}$ *is a 4-gonal K-clan if and only if* $\mathcal{O}_\alpha$ *is an oval with tangents concurrent in the point* $(0,0,1)$ *for all* $\alpha \in K^2 \setminus \{(0,0)\}$. *In this case* $\mathcal{H}_\alpha$ *is a complete arc.*
(iv) $\mathcal{C}$ *satisfies (a), (c) and (d) of Theorem 2 and* $t \mapsto y_t^2$ *is surjective if and only if* $\mathcal{H}_\alpha \setminus \{(0,1,0)\}$ *is an oval with tangents concurrent in the point* $(0,1,0)$ *for all* $\alpha \in K^2 \setminus \{(0,0)\}$. *In this case* $\mathcal{H}_\alpha$ *is a complete arc.*

Proof. (i) Suppose $\mathcal{C}$ is a partial K-clan and $t \mapsto y_t$ is injective, and let $\alpha = (a,b) \in K^2 \setminus \{(0,0)\}$. The map $t \mapsto y_t^2$ is injective; so each line of $\mathrm{PG}(2,K)$ on the point $(0,0,1)$ contains at most one further point of $\mathcal{H}_\alpha$. Since the map $t \mapsto \alpha A_t \alpha^T$ is injective, each line of $\mathrm{PG}(2,K)$ on the point $(0,1,0)$ contains at most one further point of $\mathcal{H}_\alpha$. Suppose there exist distinct $t,u,v \in K$ such that $(1, y_t^2, \alpha A_t \alpha^T)$, $(1, y_u^2, \alpha A_u \alpha^T)$ and $(1, y_v^2, \alpha A_v \alpha^T)$ are collinear. Then there exists $\lambda \in K$ such that

$$\frac{\alpha A_t \alpha^T + \alpha A_u \alpha^T}{(y_t+y_u)^2} = \frac{\alpha A_u \alpha^T + \alpha A_v \alpha^T}{(y_u+y_v)^2} = \frac{\alpha A_v \alpha^T + \alpha A_t \alpha^T}{(y_v+y_t)^2} = \lambda.$$

If $b \neq 0$, consider

$$\begin{aligned} & \frac{(x_t+x_u)\left(\alpha A_t \alpha^T + \alpha A_u \alpha^T\right)}{b^2(y_t+y_u)^2} \\ = & \frac{(x_t+x_u)\left(a^2x_t + aby_t + b^2z_t + a^2x_u + aby_u + b^2z_u\right)}{b^2(y_t+y_u)^2} \\ = & \frac{a^2(x_t+x_u)^2}{b^2(y_t+y_u)^2} + \frac{a(x_t+x_u)}{b(y_t+y_u)} + \frac{(x_t+x_u)(z_t+z_u)}{(y_t+y_u)^2} \end{aligned}$$

By Lemma 2, we have $(x_t+x_u)(z_t+z_u)(y_t+y_u)^{-2} \in \mathcal{C}_1$. Further, if $k \in \mathcal{C}_1$ then $k_1^2 + k_1 + k \in \mathcal{C}_1$ for every $k_1 \in K$, so we have

$$\frac{(x_t+x_u)\left(\alpha A_t \alpha^T + \alpha A_u \alpha^T\right)}{b^2(y_t+y_u)^2} = (\lambda/b^2)(x_t+x_u) = \lambda_\alpha(x_t+x_u) \in \mathcal{C}_1$$

where $\lambda_\alpha = \lambda/b^2$. Similarly, $\lambda_\alpha(x_u+x_v) \in \mathcal{C}_1$ and $\lambda_\alpha(x_v+x_t) \in \mathcal{C}_1$. But this contradicts the hypothesis that K is full, since $\lambda_\alpha(x_t+x_u) = \lambda_\alpha(x_u+x_v) + \lambda_\alpha(x_v+x_t)$. If $b=0$, then analogous arguments show that $(\lambda/a^2)(z_t+z_u)$, $(\lambda/a^2)(z_u+z_v)$, $(\lambda/a^2)(z_v+z_t) \in \mathcal{C}_1$, and the analogous contradiction is obtained. Thus H_α is an arc in $\mathrm{PG}(2,K)$.

Conversely, suppose that $\mathcal{H}_\alpha$ is an arc for all $\alpha \in K^2 \setminus \{(0,0)\}$. Let $t,u \in K$ with $t \neq u$. For each $\alpha \in K^2 \setminus \{(0,0)\}$, the points $(0,0,1), (1, y_t^2, \alpha A_t \alpha^T)$ and $(1, y_u^2, \alpha A_t \alpha^T)$ are not collinear, so $y_t^2 \neq y_u^2$; thus $y_t \neq y_u$ and the map $t \mapsto y_t$ is injective. For each $\alpha \in K^2 \setminus \{(0,0)\}$, the points $(0,1,0), (1, y_t^2, \alpha A_t \alpha^T)$ and $(1, y_u^2, \alpha A_u \alpha^T)$ are not collinear, so $\alpha A_t \alpha^T \neq \alpha A_u \alpha^T$ and the map $t \mapsto \alpha A_t \alpha^T$ is injective; so K is a partial K-clan.

(ii) Suppose $t \mapsto y_t$ is injective. Now let $\mathcal{C}$ be a K-clan and let $\alpha \in K^2 \setminus \{(0,0)\}$. By part (i), $\mathcal{H}_\alpha$ is an arc. Suppose there exists a point $P \in \mathrm{PG}(2,K) \setminus \mathcal{H}_\alpha$ such that $\mathcal{H}_\alpha \cup \{P\}$ is an arc. Then necessarily $P = (1,y,z)$ with $y, z \neq 0$. Since $t \mapsto \alpha A_t \alpha^T$ is surjective, there exists $t \in K$ such that $\alpha A_t \alpha^T = z$. But then $P, (0,1,0)$ and $(1, y_t^2, \alpha A_t \alpha^T)$ are collinear; a contradiction.

(iii) Let $\alpha \in K^2 \setminus \{(0,0)\}$. The point $(0,1,0)$ lies on the line $[1,0,0]$ which is a tangent, and on the lines $[a,0,1]$ for $a \in K$. For $a \in K$, the map $u \mapsto \alpha A_u \alpha^T$ is bijective if and only if there exists a unique $u \in K$ such that $\alpha A_u \alpha^T = a$ which is if and only if $[a,0,1]$ contains the unique point $(1, y_u^2, \alpha A_u \alpha^T)$ of $\mathcal{O}_\alpha$. The point $(1, y_t^2, \alpha A_t \alpha^T)$ lies on the line $[y_t^2, 1, 0]$ (which is a tangent if and only if $t \mapsto y_t^2$ is injective) and on the lines $[a y_t^2 + \alpha A_t \alpha^T, a, 1]$ for $a \in K$. For $a \in K$, the map $u \mapsto (\alpha(A_t + A_u)\alpha^T)/(y_t + y_u)^2$ is bijective if and only if there exists a unique $u \in K \setminus \{t\}$ such that $(\alpha(A_t + A_u)\alpha^T)/(y_t + y_u)^2 = a$ which is if and only if $[a y_t^2 + \alpha A_t \alpha^T, a, 1]$ contains the unique further point $(1, y_u^2, \alpha A_u \alpha^T)$ of $\mathcal{O}_\alpha$.

(iv) Let $\alpha \in K^2 \setminus \{(0,0)\}$. The point $(0,0,1)$ lies on the line $[1,0,0]$ which is a tangent, and on the lines $[a,1,0]$ for $a \in K$. For $a \in K$, the map $u \mapsto y_u^2$ is bijective if and only if there exists a unique $u \in K$ such that $y_u^2 = a$ which is if and only if $[a,1,0]$ contains the unique point $(1, y_u^2, \alpha A_u \alpha^T)$ of $\mathcal{H}_\alpha \setminus \{(0,1,0)\}$. The point $(1, y_t^2, \alpha A_t \alpha^T)$ lies on the line $[\alpha A_t \alpha^T, 0, 1]$ (which is a tangent if and only if $t \mapsto \alpha A_t \alpha^T$ is injective) and on the lines $[a y_t^2 + \alpha A_t \alpha^T, a, 1]$ for $a \in K$. For $a \in K$, the map $u \mapsto (\alpha(A_t + A_u)\alpha^T)/(y_t + y_u)^2$ is bijective if and only if there exists $u \in K$ such that $(\alpha(A_t + A_u)\alpha^T)/(y_t + y_u)^2 = a$ which is if and only if $[a y_t^2 + \alpha A_t \alpha^T, a, 1]$ contains the point $(1, y_u^2, \alpha A_u \alpha^T)$ of $\mathcal{H}_\alpha \setminus \{(0,1,0)\}$. ∎

If K is finite, then in [3] representative elements $(0,1)$ and $(1, s^{1/2})$ are used in place of all $\alpha \in K^2 \setminus \{(0,0)\}$. In that case, we see that $(0,1)A_t(0,1)^T = z_t$ and $(1, s^{1/2})A_t(1, s^{1/2})^T = x_t + s^{1/2} y_t + s z_t = f_s(t)$. Further, a scale factor of $1/(1+s^{1/2}+s)$ is included in the definition of f_s; for in that case the scale factor ensures that $f_s(1) = 1$ which is desirable so that the resulting maps are o-polynomials. Thus the family of ovals constructed in Theorem 4(iii) is the herd constructed in [3].

We remark that in the proofs of the "only if" statements in (iii) and (iv) of this theorem, in order to show that each arc has a single tangent at each point, we need the 'surjective' part property (d) of Theorem 2 for each $\alpha \in K^2 \setminus \{(0,0)\}$. Equivalently, we require the flock $\mathcal{F}(\mathcal{C})$ to be derivable with respect to each element of K (see the remarks following Theorem 3.6 in [1]). It is possible that (infinite) K-clans giving rise to flocks which are not derivable for every element of K might provide examples of complete arcs which do not have the same number of tangents at each point, in contrast to the finite case.

Further, in the proof of the "if" statement in (i) of this theorem, it is only necessary that no two further points of $\mathcal{H}_\alpha$ should be collinear with either $(0,1,0)$ or $(0,0,1)$. Equivalently, $t \mapsto y_t$ and $t \mapsto \alpha A_t \alpha^T$ are both injective. If K is finite, this means that each map is a permutation of the elements of K.

Note that our definition of partial K-clan is taken directly from [1], in particular a partial K-clan has the same cardinality as K. A weaker definition of partial K-clan is possible, as a set of (upper triangular) matrices which define a partial flock. In this case the cardinality could be less than that of K; it could even be finite. With this weaker definition and the appropriate minor changes, Theorem 4 and Theorem 5 (in the next section) still hold.

4 Arcs from 4-gonal K-clans over perfect fields

Let K be a field of characteristic 2. We say that K is *perfect* if the map $x \mapsto x^2$ is an automorphism of K. For example, each finite field $\mathrm{GF}(2^h), h \geq 1$, and the union of all such fields with h odd, is perfect.

In the case that K is perfect, a partial K-clan is such that $t \mapsto y_t$ is injective, since for $t, u \in K$ with $t \neq u$, the equation $\alpha(A_t + A_u)\alpha^T = a^2(x_t + x_u) + ab(y_t + y_u) + b^2(z_t + z_u) = 0$ has only the trivial solution $\alpha = (0,0)$ and so $y_t \neq y_u$ (else $b = a((x_t + x_u)(z_t + z_u))^{1/2}, a \neq 0$ provides non-trivial solutions).

Throughout this section we assume that K is a perfect field of characteristic 2. Using arguments similar to those in Section 3, we show that to a 4-gonal K-clan there corresponds a family of ovals in $\mathrm{PG}(2, K)$.

Theorem 5 *Let K be a perfect field of characteristic 2. Let*

$$\mathcal{C} = \left\{ A_t = \begin{pmatrix} x_t & y_t \\ 0 & z_t \end{pmatrix} \,:\, t \in K \right\}$$

be a collection of matrices with $x_t, y_t, z_t \in K$. For $\alpha \in K^2 \setminus \{(0,0)\}$ define the following sets of points in $\mathrm{PG}(2, K)$:

$$\begin{aligned} \mathcal{O}_\alpha &= \{(1, y_t^2, \alpha A_t \alpha^T) : \, t \in K\} \cup \{(0,1,0)\} \\ \mathcal{H}_\alpha &= \mathcal{O}_\alpha \cup \{(0,0,1)\}. \end{aligned}$$

(i) $\mathcal{C}$ is a 4-gonal K-clan if and only if $\mathcal{O}_\alpha$ is an oval with tangents concurrent in the point $(0,0,1)$ for all $\alpha \in K^2 \setminus \{(0,0)\}$. In this case $\mathcal{H}_\alpha$ is a complete arc.
(ii) $\mathcal{C}$ satisfies (a), (c) and (d) of Theorem 2 and $t \mapsto y_t$ is surjective if and only if $\mathcal{H}_\alpha \setminus \{(0,1,0)\}$ is an oval with tangents concurrent in the point $(0,1,0)$ for all $\alpha \in K^2 \setminus \{(0,0)\}$. In this case $\mathcal{H}_\alpha$ is a complete arc.

Proof. The proofs are analogous to those of parts (iii) and (iv) of Theorem 4. ∎

5 Subquadrangles of $\mathrm{GQ}(\mathcal{C})$

Throughout this section we assume that K is a field of characteristic 2 and that the collection of matrices $\mathcal{C} = \left\{ A_t = \begin{pmatrix} x_t & y_t \\ 0 & z_t \end{pmatrix} \,:\, t \in K \right\}$ is a 4-gonal K-clan. We now investigate some subquadrangles of the generalized quadrangle $\mathrm{GQ}(\mathcal{C})$.

Let L be a subfield of K such that if $t \in L$ then $x_t, y_t, z_t \in L$. This situation would arise if $L = K$, or if L is a proper subfield of K and each of x_t, y_t, z_t is a polynomial function with coefficients in L.

For each $\alpha \in L^2 \setminus \{(0,0)\}$, define $\mathcal{G}_{\alpha,L} = \{(x\alpha, z, y\alpha) : \, x, y, z \in L\}$. Then $\mathcal{G}_{\alpha,L}$ is a subgroup of $\mathcal{G}$ and we let

$$\begin{aligned} A_{\alpha,L}(\infty) &= A(\infty) \cap \mathcal{G}_{\alpha,L} = \{(0,0,y\alpha) : \, y \in L\} \\ A_{\alpha,L}(t) &= A(t) \cap \mathcal{G}_{\alpha,L} = \{(x\alpha, x^2 \alpha A_t \alpha^T, x y_t \alpha) : \, x \in L\} \quad \text{for } t \in L \\ A^*_{\alpha,L}(t) &= A_{\alpha,L}(t) Z \quad \text{for } t \in L \cup \{\infty\}. \end{aligned}$$

Let $\mathcal{F}_{\alpha,L} = \{A_{\alpha,L}(t) : \, t \in L \cup \{\infty\}\}$ and $\mathcal{F}^*_{\alpha,L} = \{A^*_{\alpha,L}(t) : \, t \in L \cup \{\infty\}\}$.

Theorem 6 *With the notation of this section, $(\mathcal{G}_{\alpha,L}, \mathcal{F}_{\alpha,L}, \mathcal{F}^*_{\alpha,L})$ is a Kantor family, hence $\mathcal{Q}(\mathcal{G}, \mathcal{F}, \mathcal{F}^*) = \mathcal{Q}_{\alpha,L}$ is a generalized quadrangle.*

Proof. Let $s' = t' = |L|$. In the notation of Lemma 1, $\mathcal{I}' = L \cup \{\infty\}$ and it is immediate that $(\mathcal{G}_{\alpha,L}, \mathcal{F}_{\alpha,L}, \mathcal{F}^*_{\alpha,L})$ satisfies all the hypotheses of that lemma, provided we can show that the hypothesis in part (2) holds. An element of $\mathcal{G}_{\alpha,L}$ is of the form $(x\alpha, z, y\alpha)$ for some $x, y, z \in L$. This lies in $A^*_{\alpha,L}(\infty)A_{\alpha,L}(j)$ for any $j \in \mathcal{I}'$ since

$$\begin{aligned}&(x\alpha, z, y\alpha) =\\ &(0, z + x^2\alpha A_j\alpha^T + (y\alpha + xy_j\alpha)P(x\alpha)^T, y\alpha + xy_j\alpha) \cdot (x\alpha, x^2\alpha A_j\alpha^T, xy_j\alpha).\end{aligned}$$

Also, it lies in $A^*_{\alpha,L}(i)A_{\alpha,L}(\infty)$ for any $i \in \mathcal{I}'$ since

$$(x\alpha, z, y\alpha) = (x\alpha, z, xy_i\alpha) \cdot (0, 0, y\alpha + xy_i\alpha).$$

Finally, it is an element of $A^*_{\alpha,L}(i)A_{\alpha,L}(j)$ for some $i, j \in \mathcal{I}'$ with $i \neq j$ if there exist elements $u, v, a \in L$ such that

$$\begin{aligned}(x\alpha, z, y\alpha) &= (u\alpha, v, uy_i\alpha) \cdot (a\alpha, a^2\alpha A_j\alpha^T, ay_j\alpha)\\ &= \big((u+a)\alpha, v + a^2\alpha A_j\alpha^T + (uy_i\alpha)P(a\alpha)^T, (uy_i + ay_j)\alpha\big)\end{aligned}$$

which is if there exist $u, a \in L$ such that $u + a = x$ and $uy_i + ay_j = y$. The existence (and uniqueness) of such $u, a \in L$ follows since $y_i \neq y_j$. ■

By arguments analogous to those used by Payne and Maneri [11, Theorem 1] and [9], it is immediate that, for each $\alpha \in L^2 \setminus \{(0,0)\}$, there exists a subquadrangle $S_{\alpha,L}$ of GQ($\mathcal{C}$) which is isomorphic to $\mathcal{Q}_{\alpha,L}$ and contains the points (∞) and $(0,0,0)$ of GQ($\mathcal{C}$). Each such subquadrangle is an elation generalized quadrangle with parameters $(|L|, |L|)$, with base point (∞) and with elation group $\mathcal{G}_{\alpha,L}$. Further, $\mathcal{G}_{\alpha,L} = \mathcal{G}_{\beta,L}$ if and only if $\alpha = \lambda\beta$ for some $\lambda \in L \setminus \{0\}$.

Suppose from now on that L is perfect. We investigate the subquadrangles $S_{\alpha,L}$ further.

Let $\alpha \in L^2 \setminus \{(0,0)\}$. Then $\mathcal{G}_{\alpha,L}$ is a 3-dimensional vector space over L and the map

$$\begin{aligned}\phi:\ \mathcal{G}_{\alpha,L} &\rightarrow \mathrm{PG}(2, L)\\ (x\alpha, z, y\alpha) &\mapsto (x, y, \sqrt{z})\end{aligned}$$

is an isomorphism. We call $(x, y, \sqrt{z})$ the standard coordinates for $\mathcal{G}_{\alpha,L}$. In standard coordinates, $A_{\alpha,L}(\infty)$ is the point $(0,1,0)$ and $A_{\alpha,L}(t)$ is the point $(1, y_t, \sqrt{\alpha A_t\alpha^T})$ for $t \in L$.

Theorem 7 *Let L be a perfect subfield of a field K of characteristic 2. Let $\mathcal{C} = \left\{A_t = \begin{pmatrix} x_t & y_t \\ 0 & z_t \end{pmatrix} : t \in K\right\}$ be a 4-gonal K-clan such that if $t \in L$ then $x_t, y_t, z_t \in L$. For $\alpha \in L^2 \setminus \{(0,0)\}$, define the following sets of points in* PG$(2, L)$:

$$\begin{aligned}\mathcal{O}_{\alpha,L} &= \{(1, y_t, \sqrt{\alpha A_t\alpha^T}) :\ t \in L\} \cup \{(0,1,0)\}\\ \mathcal{H}_{\alpha,L} &= \mathcal{O}_{\alpha,L} \cup \{(0,0,1)\}\end{aligned}$$

(i) $\mathcal{H}_{\alpha,L}$ is an arc in $\mathrm{PG}(2,L)$.
(ii) If $t \mapsto \alpha A_t \alpha^T$ is surjective on L for all $\alpha \in L^2 \setminus \{(0,0)\}$ then $\mathcal{H}_{\alpha,L} \setminus \{(0,1,0)\}$ is an oval with tangents concurrent in $(0,1,0)$. *In this case $\mathcal{H}_{\alpha,L}$ is a complete arc.*
(iii) If the maps $L \to L, t \mapsto \alpha A_t \alpha^T$ and $L \setminus \{t\} \to L \setminus \{0\}, u \mapsto (\alpha(A_t+A_u)\alpha^T)/(y_t+y_u)^2$, for all t, are surjective for all $\alpha \in L^2 \setminus \{(0,0)\}$ then $\mathcal{O}_{\alpha,L}$ is an oval in $\mathrm{PG}(2,K)$ *with tangents concurrent in* $(0,0,1)$.

Proof. (i) First, the line joining $(0,1,0)$ and $(0,0,1)$ contains no further point of $\mathcal{H}_{\alpha,L}$. Since $t \mapsto y_t$ is an injection, each further line on $(0,0,1)$ contains at most one further point of $\mathcal{H}_{\alpha,L}$. Also, the map $t \mapsto \sqrt{\alpha A_t \alpha^T}$ is injective so each further line on $(0,1,0)$ contains at most one further point of $\mathcal{H}_{\alpha,L}$. Finally, suppose three points of $\mathcal{H}_{\alpha,L} \setminus \{(0,1,0),(0,0,1)\}$ are collinear. Then there exist $\lambda, \mu, \nu \in L \setminus \{0\}$ and distinct $t, u, v \in L$ such that

$$(\lambda, \lambda y_t, \lambda\sqrt{\alpha A_t \alpha^T}) + (\mu, \mu y_u, \mu\sqrt{\alpha A_u \alpha^T}) = (\nu, \nu y_v, \nu\sqrt{\alpha A_v \alpha^T}).$$

Applying the inverse of the standard coordinates isomorphism, we obtain

$$(\lambda\alpha, \lambda^2 \alpha A_t \alpha^T, \lambda y_t \alpha) \cdot (\mu\alpha, \mu^2 \alpha A_u \alpha^T, \mu y_u \alpha) = (\nu\alpha, \nu^2 \alpha A_v \alpha^T, \nu y_v \alpha)$$

as elements of $\mathcal{G}_{\alpha,L}$. But this contradicts property (K1) for the Kantor family $(\mathcal{G}_{\alpha,L}, \mathcal{F}_{\alpha,L}, \mathcal{F}^*_{\alpha,L})$ since the left hand side is a non-zero element of $A_{\alpha,L}(t)A_{\alpha,L}(u)$ and the right hand side is a non-zero element of $A_{\alpha,L}(v)$.

(ii) Suppose $t \mapsto \alpha A_t \alpha^T$ is surjective on L. If there exists $P \in \mathrm{PG}(2,L)$ such that $\mathcal{H}_{\alpha,L} \cup \{P\}$ is an arc, then $P = (1,y,z)$ with $y, z \neq 0$. Let $t \in L$ be such that $z^2 = \alpha A_t \alpha^T$. But then the three points $(0,1,0)$, P and $(1, y_t, \sqrt{\alpha A_t \alpha^T})$ are collinear; a contradiction. Finally, the line $[1,0,0]$ on $(0,1,0)$ contains the unique point $(0,0,1)$ of $\mathcal{H}_{\alpha,L} \setminus \{(0,1,0)\}$. For $a \in L$, let $t \in L$ be such that $a^2 = \alpha A_t \alpha^T$ and the line $[a,0,1]$ on $(0,1,0)$ contains the unique point $(1, y_t, \sqrt{\alpha A_t \alpha^T})$ of $\mathcal{H}_{\alpha,L} \setminus \{(0,1,0)\}$.

(iii) Let $\alpha \in L^2 \setminus \{(0,0)\}$ and suppose that the maps $t \mapsto \alpha A_t \alpha^T$ and $u \mapsto (\alpha(A_t + A_u)\alpha^T)/(y_t + y_u)^2$, for all t, are surjective. The point $(0,1,0)$ lies on the line $[1,0,0]$ which is tangent to $\mathcal{O}_\alpha$ and on the lines $[a,0,1]$ for $a \in L$. For $a \in L$, since the map $t \mapsto \alpha A_t \alpha^T$ is bijective on L, there exists a unique $t \in L$ such that $a^2 = \alpha A_t \alpha^T$; so $[a,0,1]$ contains the unique point $(1, y_t, \sqrt{\alpha A_t \alpha^T})$ of $\mathcal{O}_\alpha$. Let $t \in L$. The point $(1, y_t, \sqrt{\alpha A_t \alpha^T})$ lies on the line $[y_t, 1, 0]$ which is tangent to $\mathcal{O}_{\alpha,L}$ (as $t \mapsto y_t$ is injective) and on the lines $[a y_t + \sqrt{\alpha A_t \alpha^T}, a, 1]$ for $a \in L$. For $a \in L$, since the map $u \mapsto \frac{\alpha(A_t+A_u)\alpha^T}{(y_t+y_u)^2}$ is bijective, there exists a unique $u \in L$ such that $a^2 = \frac{\alpha(A_t+A_u)\alpha^T}{(y_t+y_u)^2}$; so $[a y_t + \sqrt{\alpha A_t \alpha^T}, a, 1]$ contains the unique point $(1, y_u, \sqrt{\alpha A_u \alpha^T})$ of $\mathcal{O}_{\alpha,L}$. Thus $\mathcal{O}_{\alpha,L}$ is an oval, and each tangent passes through $(0,0,1)$. ∎

Corollary *Let K be a perfect field of characteristic* 2. *Let*

$$\mathcal{C} = \left\{ A_t = \begin{pmatrix} x_t & y_t \\ 0 & z_t \end{pmatrix} \,:\, t \in K \right\}$$

be a 4*-gonal K-clan. For $\alpha \in K^2 \setminus \{(0,0)\}$, define the following sets of points in* $\mathrm{PG}(2,K)$:

$$\begin{aligned} \mathcal{O}_\alpha = \mathcal{O}_{\alpha,K} &= \{(1, y_t, \sqrt{\alpha A_t \alpha^T}) \,:\, t \in K\} \cup \{(0,1,0)\} \\ \mathcal{H}_\alpha = \mathcal{H}_{\alpha,K} &= \mathcal{O}_{\alpha,K} \cup \{(0,0,1)\} \end{aligned}$$

Then $\mathcal{O}_\alpha$ is an oval with tangents concurrent in $(0,0,1)$, *so $\mathcal{H}_\alpha$ is a complete arc. Also, $\mathcal{H}_\alpha \setminus \{(0,1,0)$ is an oval with tangents concurrent in* $(0,1,0)$. *Further, $S_\alpha = S_{\alpha,K}$ is a generalized quadrangle isomorphic to $T_2(\mathcal{O}_\alpha)$.*

Proof. We only need to prove the last statement. In the case that $\mathcal{O}_\alpha$ is an oval, the standard isomorphism is a correspondence between the Kantor family $(\mathcal{G}_\alpha, \mathcal{F}_\alpha, \mathcal{F}^*_\alpha) = (\mathcal{G}_{\alpha,K}, \mathcal{F}_{\alpha,K}, \mathcal{F}^*_{\alpha,K})$ and the Kantor family in Example 1; so S_α is isomorphic to $T_2(\mathcal{O}_\alpha)$. ■

6 Examples

De Clerck and Van Maldeghem [4] and Bader and Payne [1] have discussed several infinite families of K-clans, most of which are generalisations of q-clans.

In the following, let K be a field of characteristic 2.

Example 2 (Classical, [4, Ex 1], [1, Ex 1]) Let $a \in K$ be such that $x^2 + x + a$ is irreducible over K and let $\mathcal{C} = \left\{ A_t = \begin{pmatrix} t & t \\ 0 & at \end{pmatrix} : t \in K \right\}$. Then $\mathcal{C}$ is a 4-gonal K-clan and for $\alpha = (\alpha_1, \alpha_2) \in K^2 \setminus \{(0,0)\}$, we obtain the oval

$$\mathcal{O}_\alpha = \left\{ \left(1, t^2, (\alpha_1^2 + \alpha_1\alpha_2 + a\alpha_2^2)t\right) : t \in K \right\} \cup \{(0,1,0)\}.$$

All these ovals are irreducible conics.

Example 3 (FTWKB [2, 6, 7, 15], [4, Ex 4], [1, Ex 2]) Suppose the map $K \to K, x \mapsto x^3$ is a bijection and let $\mathcal{C} = \left\{ A_t = \begin{pmatrix} t & t^2 \\ 0 & t^3 \end{pmatrix} : t \in K \right\}$. Then $\mathcal{C}$ is a 4-gonal K-clan (see [7, Remark 7]) and for $\alpha = (\alpha_1, \alpha_2) \in K^2 \setminus \{(0,0)\}$, we obtain the oval

$$\mathcal{O}_\alpha = \left\{ \left(1, t^4, \alpha_1^2 t + \alpha_1\alpha_2 t^2 + \alpha_2^2 t^3\right) : t \in K \right\} \cup \{(0,1,0)\}.$$

These ovals are translation ovals.

Example 4 (Payne [9], [1, Ex 4]) Let K be the union of the finite fields GF(2^h) for all odd h; so K is both full and perfect. Let $\mathcal{C} = \left\{ A_t = \begin{pmatrix} t & t^3 \\ 0 & t^5 \end{pmatrix} : t \in K \right\}$. Then $\mathcal{C}$ is a 4-gonal K-clan and for $\alpha = (\alpha_1, \alpha_2) \in K^2 \setminus \{(0,0)\}$, we obtain the oval

$$\mathcal{O}_\alpha = \left\{ \left(1, t^6, \alpha_1^2 t + \alpha_1\alpha_2 t^3 + \alpha_2^2 t^5\right) : t \in K \right\} \cup \{(0,1,0)\}.$$

Example 5 (Subiaco [3], [1, Ex 8]) Let K be the union of the finite fields GF(2^h) for all odd h; so K is both full and perfect. Let $\mathcal{C} = \left\{ A_t = \begin{pmatrix} f(t) & t^{1/2} \\ 0 & g(t) \end{pmatrix} : t \in K \right\}$ where $x^2 + dx + 1$ is irreducible over K and

$$\begin{aligned} f(t) &= \frac{d^2t^4 + d^3t^3 + (d^2 + d^4)t^2}{t^4 + d^2t^2 + 1} + \left(\frac{t}{d}\right)^{1/2} \\ g(t) &= \frac{(d^2 + d^4)t^3 + d^3t^2 + d^2t}{t^4 + d^2t^2 + 1} + \left(\frac{t}{d}\right)^{1/2} \end{aligned}$$

Then $\mathcal{C}$ is a 4-gonal K-clan and we obtain ovals as above.

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