Lusternik-Schnirelmann category of classifying spaces.

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Abstract

Let X be a finite simply connected CW-complex. In this paper, we show that the Lusternik-Schnirelmann category of the classifying space $B\,aut\,X$ is infinite if $X = S^n \vee Y$.

1 Introduction

In this paper X will denote a simply connected CW-complex of finite type, that is, $H^n(X, \mathbb{Q})$ is a finite dimensional $\mathbb{Q}$-vector space, for each n. Recall that the Lusternik-Schnirelmann category of a topological space, $cat(X)$, is the least integer n such that X can be covered by $(n+1)$ open subsets contractible in X, and is infinite if no such n exists. If H^* denotes the cohomology with any coefficient ring, we have

$$cat(X) \geq nil\, \tilde{H}^*(X), \tag{1}$$

where nil denotes the index of nilpotency of a given ring.

Let $f : X \to Y$ be a continuous map. The category of f, denoted by $cat(f)$, is the least integer n such that X is covered by $n+1$ open subsets U_1, $U_2, \cdots, U_{n+1}$ such $f_{|U_i}$ is nullhomotopic. Note that $cat(X)$ is equal to the category of the identity map, and

$$cat(f) \leq min\,\{cat(X),\, cat(Y)\}. \tag{2}$$

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Denote by X_0 the localization of X at zero, the rational Lusternik-Schnirelmann category, $cat_0(X)$, is defined by $cat_0(X) = cat(X_0)$. It verifies $cat_0(X) \leq cat(X)$ [5].

An approximation of the category of a space is given by the "mapping theorem", which states that, if $f : X \to Y$ is such that $\pi_*(f) \otimes \mathbb{Q}$ is injective, then $cat_0(X) \leq cat_0(Y)$ [5].

In this paper we will use the theory of minimal models. The Sullivan minimal model of X is a free commutative cochain algebra $(\Lambda Z,\, d)$ such that $dZ \subset \Lambda^{\geq 2} Z$. Moreover $Z^n \cong Hom_{\mathbb{Z}}(\pi_n(X), \mathbb{Q})$ [12, 9]. The Quillen minimal model of X is a free chain Lie algebra $(\mathbb{L}(V),\, \delta)$ satisfying $\delta V \subset \mathbb{L}^{\geq 2} V$ and the graded vector space V is related to the cohomology of X by $V_n \cong H^{n+1}(X, \mathbb{Q})$ [10, 1].

Fibrations of fibre in the homotopy type of X are obtained, up to fibre homotopy equivalence, as pull back of the universal fibration $X \longrightarrow B\,aut^{\bullet} X \longrightarrow B\,aut\,X$ [3, 4]; here $aut\,X$ denotes the topological monoid of all self-homotopy equivalences of X, $aut^{\bullet} X$ is the submonoid of $aut\,X$ consisting of pointed self-homotopy equivalences of X, and B is the Dold-Lashof functor [2]. Let $\tilde{B}\,aut\,X \xrightarrow{f} B\,aut\,X$ be the universal covering, the induced fibration $X \longrightarrow \tilde{B}\,aut^{\bullet} X \longrightarrow \tilde{B}\,aut\,X$ is universal for fibrations with simply connected base spaces [4, Proposition 4.2].

This work deals with the calculation of the Lusternik-Schnirelmann category of $B\,aut\,X$ under restrictions on X. The computation of $cat(B\,aut\,X)$ is of great interest as shown by the following results.

Proposition 1. *Let X be a 1-connected CW-complex and G a connected compact Lie group acting on X. If $cat_0(\tilde{B}\,aut\,X)$ is finite, then the Borel fibration $X \longrightarrow EG \times_G X \longrightarrow BG$ is rationally trivial.*

Proof. Let $f : BG \longrightarrow \tilde{B}\,aut\,X$ be the classifying map of the Borel fibration $X \longrightarrow EG \times_G X \longrightarrow BG$. Consider the map $H^*(f, \mathbb{Q}) : H^*(\tilde{B}\,aut\,X,\, \mathbb{Q}) \longrightarrow H^*(BG, \mathbb{Q}) = \Lambda V$, where V is concentrated in even degrees. Suppose now that $cat_0(\tilde{B}\,aut\,X)$ is finite. Then $H^*(f, \mathbb{Q})$ is trivial, otherwise the nilpotency index of $\tilde{H}^*(\tilde{B}\,aut\,X,\, \mathbb{Q})$ is infinite.

Suppose that $f : BG \longrightarrow \tilde{B}\,aut\,X$ is not rationally trivial. Denote by $\phi : (\Lambda W,\, d) \to (\Lambda V,\, 0)$ the Sullivan minimal model of f. Let n be the least positive integer such that $\phi(x) \neq 0$, for some $x \in W^n$. But ϕ factors through $(\Lambda W / \Lambda W^{<n},\, \bar{d})$ as $(\Lambda W,\, d) \xrightarrow{p} (\Lambda W / \Lambda W^{<n},\, \bar{d}) \xrightarrow{\bar{\phi}} (\Lambda V,\, 0)$, where p is the natural projection. But $H(\bar{\phi})$ is not trivial as $H(\bar{\phi})([x]) \neq 0$.

By the mapping theorem $cat_0(\Lambda W / \Lambda W^{<n},\, \bar{d})$ is finite. Hence $\tilde{H}^*(\bar{\phi}) = 0$, which leads to a contradiction. Therefore ϕ is the trivial map, that is, $f : BG \longrightarrow \tilde{B}\,aut\,X$ is rationally trivial. ∎

Let $X \to E \xrightarrow{p} B$ be a fibration. The genus of p, $genus(p)$, is the least integer n such that B can be covered by $n + 1$ open subsets over each of which p is a trivial fibration. The genus of p is equal to the category of the classifying map $B \to B\,aut\,X$. Hence $cat(B\,aut\,X)$ is an upper bound for the genus of any fibration of fibre X. If we put $X = K(\mathbb{Z}, 2n)$, we get that $\tilde{B}\,aut\,X$ has the rational homotopy type of $S^{2n+1}_{\mathbb{Q}}$, which is of LS category 1 (see for instance [6]). Hence we get the following

Proposition 2. *If B is simply connected, then every non trivial fibration $K(\mathbb{Z}, 2n) \to E \to B$ is of genus* 1.

Although interesting applications arise when $cat(\tilde{B}\,aut\,X)$ is finite, we do not know if such can happen when X has the rational homotopy type of a finite CW-complex. On the contrary, $cat(\tilde{B}\,aut\,X)$ is infinite in many cases (see [6, 7, 8]). Our goal is to prove that $cat(\tilde{B}\,aut\,X)$ is infinite if $X = Y \vee S^n$.

2 Models of the classifying space

A model for the classifying space $\tilde{B}\,aut\,X$ was first given by Sullivan in [12] and later by Schlessinger-Stasheff [11] and Tanré [13].

We briefly recall the construction of the model of Schlessinger-Stasheff.

Define the Lie algebra of derivations $(Der\mathbb{L}(V), D)$ as follows: $Der\mathbb{L}(V) = \bigoplus_{k\geq 1} Der_k(\mathbb{L}(V))$, where $Der_k(\mathbb{L}(V))$ is the vector space of derivations which increase the degree by k, with the restriction that $Der_1(\mathbb{L}(V))$ is the vector space of derivations of degree one which commute with the differential δ.

Given two derivations θ and θ', the Lie bracket is defined by $[\theta, \theta'] = \theta\theta' - (-1)^{|\theta||\theta'|}\theta'\theta$ and the differential D is defined by $D\theta = [\delta,\, \theta]$.

Define the differential Lie algebra $(s\mathbb{L}(V) \underset{\sim}{\oplus} Der\mathbb{L}(V),\, \tilde{D})$ as follows:

- $s\mathbb{L}(V) \underset{\sim}{\oplus} Der\mathbb{L}(V)$ is isomorphic to $s\mathbb{L}(V) \oplus Der\mathbb{L}(V)$ as a graded vector space,
- If $\theta, \theta' \in Der\,\mathbb{L}(V)$ and $sx, sy \in s\mathbb{L}(V)$, then $[\theta, \theta'] = \theta\theta' - (-1)^{|\theta||\theta'|}\theta'\theta$, $[\theta, sx] = (-1)^{|\theta|}\, s\theta(x)$, $[sx, sy] = 0$,
- $\tilde{D}(\theta) = [\delta, \theta], \tilde{D}(sx) = -s\delta x + ad\,x$, where $ad\,x$ is the derivation of $\mathbb{L}(V)$ defined by $(ad\,x)(y) = [x, y]$.

Theorem 3. [11, 13] *A model of the universal fibration $X \longrightarrow \tilde{B}\,aut^{\bullet}X \longrightarrow \tilde{B}\,aut\,X$ is given by*

$$(\mathbb{L}(V), \delta) \longrightarrow (Der\mathbb{L}(V), D) \longrightarrow (s\mathbb{L}(V) \underset{\sim}{\oplus} Der\mathbb{L}(V), \tilde{D}).$$

A model of $\tilde{B}\,aut\,X$ from derivations of the Sullivan minimal model of X is described in [12].

We will suppose henceforth that X is a finite simply connected CW-complex. We know that the LS category of $B\,aut\,X$ is not finite in various cases, among them when X is an elliptic space (i.e. $\pi_*(X)\otimes\mathbb{Q}$ is finite dimensional), a wedge of spheres or a product space $X = Y \times Z$ [6, 7].

One may expect, by duality, the LS-category of $B\,aut\,X$ to be infinite when $X = Y \vee Z$. We show that it is the case if Z is a wedge of spheres.

3 The theorem

Theorem 4. *The Lusternik-Schnirelmann category of $B\,aut\,X$ is infinite if $X = Y \vee Z$, where Z is a wedge of spheres.*

Proof of the theorem

Case 1: $X = Y \vee S^{2n}$.

Let $F \longrightarrow E \longrightarrow B$ be a fibration, then $cat(E) \leq (cat(B)+1).(cat(F)+1)-1$. Applying this to the universal fibration $X \longrightarrow B\ aut^\bullet\ X \longrightarrow B\ aut\ X$, we get $cat(B\ aut^\bullet X) \leq (cat(B\ aut\ X)+1).(cat(X)+1)-1$.

As $cat(X)$ is finite, we deduce that $cat(BautX)$ is infinite whenever $cat(Baut^\bullet X)$ is infinite.

The Quillen minimal model of X is $(\mathbb{L}(V), \delta) = (\mathbb{L}(W \oplus \mathbb{Q}.x_{2n-1}), \delta)$ where $\delta(x_{2n-1}) = 0$ and $\delta(W) \subset \mathbb{L}(W)$.

Let θ be the derivation defined by $\theta(x_{2n-1}) = [x_{2n-1}, x_{2n-1}]$, $\theta(W) = 0$. Let us show that θ is a cycle in $(Der\mathbb{L}(V), D)$. Obviously $[\delta, \theta](x_{2n-1}) = 0$ and if $w \in W$, then $[\delta, \theta](w) = \delta\theta(w) + \theta(\delta w) = \theta(\delta w)$. But $\delta(w) \in \mathbb{L}(W)$, therefore $\theta(\delta w) = 0$. Moreover, θ cannot be a boundary. If it is, then there exists a derivation θ' such that $[\delta, \theta'](x_{2n-1}) = \delta\theta'(x_{2n-1}) = \theta(x_{2n-1}) = [x_{2n-1}, x_{2n-1}]$; what should imply that $[x_{2n-1}, x_{2n-1}]$ is a boundary in $(\mathbb{L}(V), \delta)$.

As $[\theta, \theta] = 0$, the injection of the Lie subalgebra generated by θ provides a morphism $K(\mathbb{Q}, 2n) \longrightarrow (\tilde{B}\ aut^\bullet\ X)_0$ that induces an injective map in homotopy. Therefore, applying the mapping theorem [5], $cat(\tilde{B}\ aut^\bullet\ X)$ is infinite.

Case 2: $X = Y \vee S^{2n+1}$.

The Quillen minimal model of X is $(\mathbb{L}(V), \delta) \amalg \mathbb{L}(x, 0)$ with $|x| = 2n$.

1. Suppose that $H_{even}(\mathbb{L}(V), \delta) = 0$ and let $[\alpha] \in H_q(\mathbb{L}(V), \delta)$ where q is odd. Define a sequence of derivations θ_n of $(\mathbb{L}(V), \delta) \amalg \mathbb{L}(x, 0)$ by $\theta_n(V) = 0, \quad \theta_n(x) = \underbrace{[\alpha, [\alpha, \cdots, [\alpha}_{2n}, x] \cdots]$, $n \geq 1$. The derivation θ_n is a cycle but cannot be a boundary. Moreover, $[\theta_m, \theta_n] = 0$. Therefore $\{\theta_n\}_{n \geq 1}$ generate an abelian Lie algebra, which we denote by $Ab(\theta_n,\, n \geq 1)$. The inclusion $Ab(\theta_n,\, n \geq 1) \to Der(\mathbb{L}(V) \amalg \mathbb{L}(x))$ induces an injective map in homology, hence the corresponding mapping
$$\Pi_{n \geq 1} S^{2n|\alpha|+1} \to \tilde{B}\, aut^\bullet\, X$$
induces an injective map in rational homotopy.

2. Suppose that $H_{even}(\mathbb{L}(V), \delta) \neq 0$. Take $[\beta] \in H_q(\mathbb{L}(V), \delta)$ where q is even. For each $n \geq 1$, define a derivation γ_n of $(\mathbb{L}(V), \delta) \amalg \mathbb{L}(x, 0)$ by $\gamma_n(V) = 0, \quad \gamma_n(x) = \underbrace{[\beta, [\beta, \cdots, [\beta}_{n}, x] \cdots]$ and argue as in the previous case.

3. If $\tilde{H}_*(\mathbb{L}(V), \delta) = 0$, then X has the rational homotopy type of S^{2n+1}. A direct computation shows that $(\tilde{B}\, aut\, X)_0$ has the rational homotopy type of $K(\mathbb{Q}, 2n+2)$.

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