Holomorphic Cliffordian product

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Abstract

Let $\mathbb{R}_{0,n}$ be the Clifford algebra of the antieuclidean vector space of dimension n. The aim is to built a function theory analogous to the one in the $\mathbb{C}$ case. In the latter case, the product of two holomorphic functions is holomorphic, this fact is, of course, of paramount importance. Then it is necessary to define a product for functions in the Clifford context. But, non-commutativity is inconciliable with product of functions. Here we introduce a product which is commutative and we compute some examples explicitely.

1 Introduction

In one complex variable, it is possible to define a product of two holomorphic functions f and g by $(fg)(z) = f(z)g(z)$ because this last expression is holomorphic. Here we make use of commutativity and of Cauchy-Riemann equations which are first order partial differential equations. But in fact, there is much more than that. Holomorphy is equivalent of analyticity : taking $f(z) = \Sigma a_p z^p$ and $g(z) = \Sigma a_q z^q$ then

$$(fg)(z) = \sum_n \Big(\sum_{p+q=n} a_p \; b_q \Big) z^n.$$

We can do the product either in the space of the values or in the space of the variable and parameters. For higher dimensional spaces, in Clifford analysis, the above two possibilities give two different results. The first product is

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useless because if $f(x)$ and $g(x)$ are monogenic [1], [3], or regular [6], or holomorphic Cliffordian [9], $f(x)g(x)$ is not. In [1], F. Bracks, R. Delanghe, F. Sommen defined the Cauchy Kovalewski product, but it is no so easy to work with it [8]. The existence of a product is one of the principal questions in Clifford analysis, see [11] and [13]. In [7] D. Hestenes and G. Sobczyk defined the inner product. In [10] H. Malonek worked with his permutational product. It is related to Fueter's ideas [5].

The anticommutator $\{a,b\} = 1/2(ab+ba)$ is well known, but when we have three elements, we get $\{a,\{b,c\}\}$ or $\{\{a,b\},c\}$ or $\{\{a,c\},b\}$. In several papers [12], [14], F. Sommen uses the basic fact that the anticommutator of two vectors is a scalar and hence commutes with all elements. By the same token here a basic fact is that the anticommutator of two paravectors is a paravector.

In quantum mechanics other products are defined : chronological product, normal order in product.

Notations.

Let $\mathbb{R}_{0,n}$ the Clifford algebra of the real vector space V of dimension n, provided with a quadratic form of negative signature. Denote by S the set of scalars in $\mathbb{R}_{0,n}$ which can be identified to $\mathbb{R}$. An element of the vector space $S \oplus V$ is called a paravector. Let $\{e_i\}, i = 1, \ldots, n$ be an orthonormal basis of V and let $e_0 = 1$. We have $e_ie_j + e_je_i = -2\delta_{ij}$ for $1 \leq i,j \leq n$. On $S \oplus V$ we have two quadratic structures : one with signature $+ - \cdots -$, the other with signature $+ + \cdots +$. In this latter case the scalar product is denoted by $(a \mid b)$. To do analysis, we take a norm on $S \oplus V$ such that $\|ab\| \leq \|a\| \, \|b\|$.

For any paravector u, we split up the real part u_0 and the vectorial part $\vec{u}$:

$$u = u_0 + \vec{u}.$$

2 Algebraic structure on the paravector space

2.1 Symmetric product

Theorem and definition 1.- For $\ell \in \mathbb{N} \setminus \{0\}$ define the multilinear symmetric function

$$\begin{aligned} E:\ (S \oplus V)^\ell &\longrightarrow\ S \oplus V \\ (u_1, \ldots, u_\ell) &\longrightarrow\ \frac{1}{\ell!} \sum_{\sigma \in \mathfrak{S}_\ell} u_{\sigma(1)} \ldots u_{\sigma(\ell)} \end{aligned}$$

where $\mathfrak{S}_\ell$ is the set of all permutations of $\{1, \ldots, \ell\}$.

Proof.- It is obvious that this function is multilinear and symmetric. To prove that the values are in $S \oplus V$, we need a lemma, but before stating it, it is useful to introduce an algorithmic symbol :

$$(1) \qquad € \prod_{i=1}^{\ell} u_i € \; := \frac{1}{\ell!} \sum_{\sigma \in \mathfrak{S}_\ell} u_{\sigma(1)} \dots u_{\sigma(\ell)}.$$

It is easier to work with this than with $E(u_1, \dots, u_\ell)$.

Lemma 1.

$$(2) \quad € \prod_{i=1}^{\ell} u_i € \; = \frac{1}{\ell} \sum_{i=1}^{\ell} u_i € \prod_{\substack{j=1 \\ j \neq i}}^{\ell} u_j € \; = \frac{1}{\ell} \sum_{i=1}^{\ell} € \prod_{\substack{j=1 \\ j \neq i}}^{\ell} u_j € \, u_i =$$
$$\frac{1}{2\ell}\Big(\sum_{i=1}^{\ell} u_i \; € \prod_{\substack{j=1 \\ j \neq i}}^{\ell} u_j € + \sum_{i=1}^{\ell} \; € \prod_{\substack{j=1 \\ j \neq i}}^{\ell} u_j € \; u_i \Big).$$

Proof of (2).- The first and second formulas are factorisations of the symmetric product. The third one is a mean of these two.

Now, to prove that the values of the function E is in $S \oplus V$, we use induction on ℓ.

For $\ell = 1$ the result is trivial, for $\ell = 2$, we have

$$€ \, ab \, € \; = \frac{1}{2}(ab + ba)$$

is in $S \oplus V$. The last formula (2) allows us to finish the recurrence.

Proposition 1.- *For* $i = 1, \dots, \ell$ *and* $u_i \in S \oplus V$

$$(3) \qquad \| € \prod_{i=1}^{\ell} \; u_i € \| \; \leq \; \| \prod_{i=1}^{\ell} u_i \|.$$

This follows from the definition.

Extension of the symbol.

Let φ be a linear function :

$$\begin{aligned} (S \oplus V)^k &\longrightarrow (S \oplus V)^\ell \\ (u_1, \dots, u_k) &\longrightarrow \big(\varphi_1(u_1, \dots, u_k), \dots, \varphi_\ell(u_1, \dots, u_k)\big) \end{aligned}$$

then, we define

$$(4) \qquad € \prod_{i=1}^{\ell} \varphi_i(u_1, \dots, u_p) € \; := E \circ \varphi \; (u_1, \dots, u_p).$$

Remark 1.

It is always possible to restrict the symmetrization to only $n = \dim V$ factors because if we have ℓ paravectors $u_1, \dots, u_\ell$ we can take $\vec{u}_{i_1}, \dots, \vec{u}_{i_p}$, p linearly independent vectors, all paravectors u_j are linear combinations of 1 and the $\vec{u}_{i_k}$ and the symmetrization is on $\vec{u}_{i_1}, \dots, \vec{u}_{i_p}$.

Remark 2.

€ $\prod_{i=1}^{k} A_i$ € is well defined for all $A_i \in \mathbb{R}_{0,n}$ because A_i are sums and products of paravectors and we have linearity.

Remark 3.

€ x € yz € € makes sense but it is clumsy and it is a pitfall, so we shall avoid using it. In general it is not equal to € xyz € .

2.2 The symmetric algebra of V

For $n = 1$, $\mathbb{R}_{0,1} = \mathbb{C}$ we have a special phenomena. Take $\mathbb{R}[X]$ the algebra of polynomials in one indeterminate, then $\mathbb{R}[X]/(X^2+1)$ is $\mathbb{C}$. But $\mathbb{R}[X_1, \dots, X_n]$, the algebra of polynomials in n indeterminates is not directly connected with $\mathbb{R}_{0,n}$. This algebra of polynomials is clearly built to do products.

Inside the € we compute in $\mathbb{R}[e_1, \dots, e_n]$ which may be identified with the symmetric algebra (algebra of symmetric tensors) of the vector space V.

2.3 Examples

In the following formulas a, b, c are in $S \oplus V$.

$$
\begin{aligned}
\text{€}\ a\ \text{€} &= a \\
\text{€}\ ab\ \text{€} &= \frac{1}{2}(ab + ba) \\
\text{€}\ a^2 b\ \text{€} &= \frac{1}{3}(a^2 b + aba + ba^2).
\end{aligned}
$$

It is important to notice that this is not $\frac{1}{2}\ (a^2 b + ba^2)$

$$
\begin{aligned}
\text{€}\ e_1^2 a\ \text{€} &= -\frac{2}{3} a + \frac{1}{3}\ e_1\ a e_1 \\
\binom{p+q}{q}\ \text{€}\ a^p b^q\ \text{€} &= \frac{d^q}{dt^q}\Big|_{q=0} (a + tb)^{p+q}
\end{aligned}
$$

Here are explicit formulas for € $\prod_{h=1}^{H} e_{i_h}$ € :

If $H \equiv 0 \pmod 4$ and if all indices are equal then it is equal to 1, otherwise it is 0.

If $H \equiv 1 \pmod 4$ and if all indices are equal then it is equal to e_{i_1}, otherwise if all indices are equal but one, say i_1, it is $\frac{1}{H} e_{i_1}$ else it is 0.

If $H \equiv 2 \pmod 4$ and if all indices are equal then it is equal to -1 otherwise 0.

If $H \equiv 3 \pmod 4$ and if all indices are equal then it is equal to $-e_{i_1}$ otherwise if all indices are equal but one, say i_1, it is $-\frac{1}{H} e_{i_1}$ else it is 0.

Proof of these values :

Take $v = (t_1 \ e_{i_1} + \ldots + t_H \ e_{i_H})$ where $t_1, \ldots, t_H$ are scalars.

Beside the coefficient, the value of the product is the homogeneous term corresponding to $t_1 t_2 \ldots t_H$ in v^H.

First case : all indices are equal, say e_{i_1}

$$\begin{aligned} v &= (t_1 + \ldots + t_H) e_{i_1} \\ v^H &= (t_1 + \ldots + t_H)^H \ e_{i_1}^H \end{aligned}$$

and $e_{i_1}^H$ is 1 or e_{i_1} or -1 or $-e_{i_1}$.

Second case : all indices but one are equal, say e_{i_1}.

$$v = t_1 \ e_{i_1} + (t_2 + \ldots + t_H) \ e_{i_2}$$

$$v^H = \begin{cases} \left(t_1^2 + (t_2 + \ldots + t_H)^2\right)^{H/2} & \text{we get } 0 \\ \left(t_1^2 + (t_2 + \ldots + t_H)^2\right)^{(H-1)/2} v & \text{we get } e_{i_1}/H \\ -\left(t_1^2 + (t_2 + \ldots + t_H)^2\right)^{H/2} & \text{we get } 0 \\ -\left(t_1^2 + (t_2 + \ldots + t_H)^2\right)^{(H-1)/2} v & \text{we get } -e_{i_1}/H \end{cases}$$

Third case : at least three different indices

$$v = t_1 \ e_{i_1} + t_2 \ e_2 + w$$

with w orthogonal to e_{i_1} and e_{i_2}

$$v^H = \begin{cases} (t_1^2 + t_2^2 + w^2)^{H/2} \\ (t_1^2 + t_2^2 + w^2)^{(H-1)/2} \ v \\ -(t_1^2 + t_2^2 + w^2)^{H/2} \\ -(t_1^2 + t_2^2 + w^2)^{(H-1)/2} \ v \end{cases}$$

We get 0 (no homogenous factor in $t_1 \ldots t_H$).

2.4 Symmetrization by integral means

The main problem of the $\in$ algorithm is the disentangling, that is to translate from $\in$ expression $\in$ to an expression without $\in$ $\in$ using the classical product in the Clifford algebra. A tool for that is Dirichlet means, which was studied extensively by B.C. Carlson [2] in a completely different situation. He uses these means for classical special functions.

Let $E_{\ell-1}$ be the standard simplex.

$$E_{\ell-1} := \{(t_1, \ldots, t_{\ell-1}) \in \mathbb{R}^{\ell-1} : \ \forall j,\ t_j \geq 0,\ \sum_{p=1}^{\ell-1} t_p \leq 1\}.$$

The beta function in ℓ variables is

$$B(b_1, \ldots, b_\ell) := \int_{E_{\ell-1}} t_1^{b_1-1} \ldots t_{\ell-1}^{b_{\ell-1}-1}(1 - t_1 - \ldots - t_{\ell-1})^{b_\ell-1}\ dt_1 \ldots dt_{\ell-1}$$

$B(b) = B(b_1, \ldots, b_\ell)$ is symmetric. For $b_j \in \mathbb{C}$, $Re\ b_j > 0$ and g integrable, the Dirichlet measure μ_b is defined by

$$(5) \quad \int_E g(t)\ d\mu_b(t) := \int_{E_{\ell-1}} g(t_1, \ldots, t_{\ell-1}) \frac{1}{B(b)}\ t_1^{b_1-1} \ldots t_{\ell-1}^{b_{\ell-1}-1}(1 - t_1 - \ldots - t_{\ell-1})^{b_\ell-1}\ dt_1 \ldots dt_{\ell-1}.$$

Definition.- For $f : S \oplus V \longrightarrow S \oplus V$ continuous and $u_1, \ldots, u_\ell$ in $S \oplus V$, put

$$F(f, b, u) := \int_E f(t\!:\!u)\ d\mu_b(t) \tag{6}$$

with $t\!:\!u := \sum_{i=1}^{\ell-1} t_i u_i + \Big(1 - \sum_{i=1}^{\ell-1} t_i\Big) u_\ell.$

This integral gives the symmetrization.

A simple illustration with two paravectors u, v

$$F(t \to t^2, 1, 1, u, v) = \int_0^1 (tu + (1-t)v)^2 dt = \frac{1}{3}u^2 + \frac{1}{3} \text{€}\ uv\ \text{€}\ + \frac{1}{3}v^2.$$

By the remark 1 of paragraph 3, it is always possible to take only simplices of dimension less than or equal to n.

3 Analysis with the holomorphic cliffordian product

3.1 Holomorphic cliffordian functions

In this paragraph, we recall some notions from [9].

Let D denote the differential operator

$$D = \sum_{i=0}^{n} e_i \frac{\partial}{\partial x_i}$$

and let Δ be the standard Laplacian

$$\Delta = \sum_{i=0}^{n} \frac{\partial^2}{\partial x_i^2}.$$

If n is odd, say $n = 2m + 1$, the vector space $\mathcal{V}$ of holomorphic cliffordian functions was defined to be the kernel ot the $D\Delta^m$ operator.

Let $x := x_0 + \sum_{i=1}^{n} e_i x_i$, it is holomorphic cliffordian as well as its powers x^k (with $k \in \mathbb{Z}$). More generally, put $\alpha := (\alpha_0, \ldots, \alpha_n)$ a multiindice, $\alpha_i \in \mathbb{N}$, and

$$|\alpha| := \sum_{i=0}^{n} \alpha_i$$

$$P_\alpha(x) := \sum_{\sigma \in \mathfrak{S}} \prod_{\nu=1}^{|\alpha|-1} \Big(e_{\sigma(\nu)}\ x\Big)\ e_{\sigma(|\alpha|)}$$

where $\mathfrak{S}$ is the permutation group with $|\alpha|$ elements. By the same token, put

$$\beta := (\beta_0, \ldots, \beta_n), \quad \beta_i \in \mathbb{N}$$

$$|\beta| \ := \sum_{i=1}^{n} \beta_i$$

$$S_\beta(x) := \sum_{\sigma \in \mathfrak{S}} \prod_{\nu=1}^{|\beta|} \Big(x^{-1}\ e_{\sigma(\nu)}\Big) x^{-1}.$$

The functions P_α and S_β are, for n odd, holomorphic cliffordian but they make sense for all n.

Recall from [9] that, when n is odd there is a Laurent type expansion for holomorphic cliffordian functions with a pole at the origin :

$$f(x) = \sum_{|\beta| < B} S_\beta(x) d_\beta + \sum_{|\alpha|=1}^{\infty} P_\alpha(x) c_\alpha$$

where, in general, d_β and c_α belong to $\mathbb{R}_{0,n}$.

The basic idea is that we work with functions which are limits of sums of x^k and their scalar derivatives. Functions generated in this manner are well-defined for all n. The problem of building a product is not connected directly with the $D\Delta^m$ operator.

First we extend the product defined in the previous part.

3.2 Extension of the product to normally convergent series

THEOREM 2.- Let $\sum_{n=0}^{\infty} a_n$ be a series which converges in norm and such that the coefficients are products of paravectors. Then the series $\sum_{n=0}^{\infty} \text{€}\ a_n\ \text{€}$ converges and

$$\text{€} \sum_{n=0}^{\infty} a_n\ \text{€} = \sum_{n=0}^{\infty} \text{€}\ a_n\ \text{€}\ .$$

Proof.- From the inequality (3)

$$\sum_{n=0}^{N} \| \text{€}\ a_n\ \text{€} \| \ \leq\ \sum_{n=0}^{N} \|a_n\|$$

thus the series $\sum_{n=0}^{\infty}$ € a_n € is convergent in norm.

By linearity

$$\text{€} \sum_{n=0}^{N} a_n\ \text{€}\ - \sum_{n=0}^{N}\ \text{€}\ a_n\ \text{€}\ = 0$$

and it suffices to let $N \to \infty$.

Now it is easy to extend the product to rational functions. First an example. We define, for $\|1-a\| < 1$

$$\text{€}\ a^{-1}b\ \text{€}\ :=\ \text{€}\ \Big(1-(1-a)\Big)^{-1} b\ \text{€}\ = \\ \text{€}\ \sum_{k=0}^{\infty}(1-a)^k b\ \text{€}\ = \sum_{k=0}^{\infty}\ \text{€}\ (1-a)^k b\ \text{€}\ .$$

In general we define, for $\|1-v_j\| < 1$

$$\text{€}\ \prod_{i=1}^{k} u_i\ \prod_{j=1}^{\ell} v_j^{-1}\ \text{€}\ := \sum_{k_1=1}^{\infty} \ldots \sum_{k_\ell=1}^{\infty}\ \text{€}\ \prod_{i=1}^{k} u_i\ \ \prod_{j=1}^{\ell}(1-v_j)^{k_j}\ \text{€}\ .$$

Of course we have to find the analytic extension for that symbol.

A classical example is the following : for $u, v \in (S \oplus V) \setminus \{0\}$

€ $u^{-1}v^{-1}$ € is defined by :

if u and v are linearly dependent with $v = \lambda u$ for some $\lambda \in \mathbb{R} \setminus \{0\}$ then it is € $u^{-1}(\lambda u)^{-1}$ € $= \lambda^{-1}\ u^{-2}$.

If u and v are linearly independant for all $t \in [0,1]$, $tu + (1-t)v$ has an inverse and we have

$$\text{€}\ u^{-1}v^{-1}\ \text{€}\ = \int_0^1 \Big(tu + (1-t)v\Big)^{-2} dt = F(t \to t^{-1}, 1, 1, u, v).$$

This was introduced in quantum mechanics by R.P. Feynmann [4].

For a proof, in the open set $\|1-u\| < 1$, $\|1-v\| < 1$ expand in series.

In general, with the hypothesis of linear independence of v_j

$$\text{(7)} \qquad \text{€}\ \prod_{i=1}^{\ell} u_i \prod_{j=1}^{\ell+1} v_j^{-1}\ \text{€}\ = \frac{1}{\ell!} \sum_{\sigma \in \mathfrak{S}_\ell} \int_E \prod_{j=1}^{\ell} \Big((t\!:\!v)^{-1} u_{\sigma(j)}\Big)\ (t\!:\!v)^{-1}\ dt_1 \ldots dt_\ell.$$

We have one more v_j than u_i. If it is not true, add some $v_j = 1$.

Remark.- Inside the € we compute in the field of fractions of $\mathbb{R}[e_1, \ldots, e_n]$.

3.3 Integral representation formulas for holomorphic Cliffordian products

The standard spectral theory allows us to write

$$f(A) = \frac{1}{2i\pi} \oint f(z) \frac{1}{z-A} \, dz.$$

In particular

$$A^n = \frac{1}{2i\pi} \oint z^n \frac{1}{z-A} \, dz.$$

Now, let u_1 and u_2 be linearly independant elements of the vector space V, then

$$\in u_1^p \; u_2^q \in \; = \frac{1}{(2i\pi)^2} \oint_{C_1} \oint_{C_2} z_1^p z_2^q \int_0^1 \Big(t(z_1 - u_1) + (1-t)(z_2 - u_2) \Big)^{-2} \, dt \; dz_1 \; dz_2.$$

Where C_1 and C_2 are positively oriented simply closed contours, such that the eigenvalues are inside these contours.

For $u \in S \oplus V$ with $u = u_0 + \vec{u}$, the eigenvalues are $u_0 \pm i \, \|\vec{u}\|$

For a general integral representation formula, it is possible to reduce to the case where $\{u_1, \ldots, u_\ell\}$ are paravectors and are linearly independent, then formally :

(8) $\quad \in f(u_1, \ldots, u_\ell) \in \; =$

$$= \frac{1}{(2i\pi)^\ell} \oint_{C_1} \cdots \oint_{C_\ell} f(z_1, \ldots, z_\ell) \; F(t \to t^{-\ell}, 1, \ldots, 1, z_1 - u_1, \ldots, z_\ell - u_\ell) dz_1 \ldots dz_\ell.$$

3.4 Interpolation by polynomials

Theorem 3.- The interpolation formula of Lagrange. Let $x_0, \ldots, x_\ell$, $\ell+1$ paravectors, $a_0, \ldots, a_\ell$, $\ell+1$ paravectors. Put

$$P(x) := \sum_{i=0}^{\ell} \in a_i \prod_{\substack{k \neq i \\ k=0}}^{\ell} \frac{x - x_k}{x_i - x_k} \in . \tag{9}$$

Then, for all $j = 0, \ldots, \ell$, $P(x_j) = a_j$ and, for n odd, P is an holomorphic Cliffordian polynomial of degre ℓ.

Proof.-

$$P(x_j) = \in a_j \prod_{\substack{k \neq j \\ k=0}}^{\ell} \frac{x_j - x_k}{x_j - x_k} \in \; = a_j.$$

The desentangling is easy. Put

$$\alpha_i = \sum_{\substack{k=0 \\ k \neq i}}^{\ell} t_k (x_i - x_k) + t_i + \Big(1 - \sum_{k=0}^{\ell} t_k \Big)$$

$$\beta_{k,i} = \begin{cases} x - x_k & \text{if } k \neq i \\ a_i & \text{if } k = i \end{cases}$$

Then

$$P(x)=\sum_{i=0}^{\ell}\frac{1}{(\ell+1)!}\sum_{\sigma\in\mathfrak{S}_{\ell+1}}\int_{E_\ell}\prod_{k=0}^{\ell}\left(\alpha_i^{-1}\beta_{\sigma(k),i}\right)\alpha_i^{-1}\,dt_0\,dt_1\,\ldots\,dt_\ell. \tag{10}$$

where $\mathfrak{S}_{\ell+1}$ is the permutation group of $\{0,1,\ldots,\ell\}$. This formula shows that P is holomorphic Cliffordian in x but also in x_k and a_k.

3.5 Product of holomorphic cliffordian functions

From the point of view of the product, the $S_\beta(x)$ are natural :

put

$$\begin{aligned}
\partial^\beta &:= \frac{\partial^{\beta_0+\cdots+\beta_n}}{\partial x_0^{\beta_0}\cdots\partial x_n^{\beta_n}}\\
\text{€}\ S_\beta(x)\ \text{€}\ &= \text{€}\ (-1)^{|\beta|}\ \partial^\beta x^{-1}\ \text{€}\\
&= (-1)^{|\beta|}\ \partial^\beta\ \text{€}\ x^{-1}\ \text{€}\\
&= (-1)^{|\beta|}\ \partial^\beta x^{-1}\\
&= S_\beta(x).
\end{aligned}$$

But the $P_\alpha(x)$ are, in general, different from $\text{€}\ P_\alpha(x)\ \text{€}$. For example :

$$\text{€}\ e_1^2 x\ \text{€}\ = \frac{1}{3}\ e_1\ x\ e_1-\frac{2}{3}\ x.$$

Let

$$k_\alpha := \frac{|\alpha|\ !}{\alpha_0\ !\ \ldots\ \alpha_n\ !}$$

we have

$$\text{€}\ P_\alpha(x)\ \text{€}\ = k_\alpha\ \partial^\alpha\ x^{2|\alpha|-1}$$

because the left side is

$$\text{€}\ P_\alpha(x)\ \text{€}\ =\ |\alpha|\ !\ \text{€}\ e_0^{\alpha_0}\ldots e_n^{\alpha_n}\ x^{|\alpha|-1}\ \text{€}$$

and the right side is

$$\begin{aligned}
\partial^\alpha\ x^{2|\alpha|-1} &= \text{€}\ \partial^\alpha\ x^{2|\alpha|-1}\ \text{€}\\
&= \alpha_0!\ldots\alpha_n!\ \text{€}\ e_0^{\alpha_0}\ldots e_n^{\alpha_n}\ \ x^{|\alpha|-1}\ \text{€}.
\end{aligned}$$

We may conclude that the set of polynomials $\partial^\alpha\ x^k$, $k\in\mathbb{N}$ are better.

For h and k in $\mathbb{N}$, let

$$\begin{aligned}
p(x) &= \text{€}\ e_0^{\alpha_0}\cdots e_n^{\alpha_n}\ x^h\ \text{€}\\
q(x) &= \text{€}\ e_0^{\beta_0}\cdots e_n^{\beta_n}\ x^k\ \text{€}.
\end{aligned}$$

Then, their product is

$$\text{€}\ p(x)q(x)\ \text{€}\ =\ \text{€}\ e_0^{\alpha_0+\beta_0}\cdots e_n^{\alpha_n+\beta_n}\ x^{h+k}\ \text{€}.$$

Here are other examples of products of holomorphic cliffordian functions.

Product of the exponential and a constant :

$$\begin{aligned} \text{€}\ a e^{x}\ \text{€}\ &= \int_0^1 \ e^{tx}\ a\ e^{(1-t)x}\ dt \\ &= \frac{d}{ds}\Big|_{s\,=\,0} \quad e^{x+sa}. \end{aligned}$$

Product of two exponentials :

$$\text{€}\ e^{x}\ e^{y}\ \text{€}\ = \text{€}\ e^{x+y}\ \text{€}\ = e^{x+y}.$$

Product of rational functions :

$$\text{€}\ \frac{a}{x-b}\ \text{€} = \frac{d}{ds}\Big|_{s\,=\,0} \quad \int_0^1 \ \Big(t+(1-t)(x-b)+sa\Big)^{-1} ds$$

$$\text{€}\ \frac{1}{(x-a)^{p}\ (x-b)^{q}}\ \text{€}\ = \frac{(p+q+1)\ !}{(p-1)\ !\ (q-1)\ !} \int_0^1 \ \Big(ta+(1-t)b\Big)^{-(p+q+2)} \quad t^{p}(1-t)^{q} dt.$$

The computations are the usual ones, by example :

$$\text{€}\ \frac{1}{x-a} - \frac{1}{x-b}\ \text{€}\ = \ \text{€}\ \frac{a-b}{(x-a)(x-b)}\ \text{€}$$

this means

$$(x-a)^{-1}-(x-b)^{-1} = \int_0^1 \ \Big(x-(ta+(1-t)b)\Big)^{-1} \ (a-b)\ \Big(x-(ta+(1-t)b)\Big)^{-1} \ dt.$$

The basic fact is that there is no difference between “variable” and “constants” : for n odd, all expressions are holomorphic cliffordian with respect to their constants too.

3.6 Derivatives and equations of Cauchy-Riemann type

For $u \in S \oplus V,\ u = \sum_{j=0}^{n} u_j e_j,$ the directional derivative is

$$(u \mid \nabla_x) := \sum_{j=0}^{n} \ u_j \ \frac{\partial}{\partial x_j}.$$

Lemma 2.- Let $u \in S \oplus V$, $a \in \mathbb{R}_{0,n}$ $p \in \mathbb{N}$, then

$$(u \mid \nabla_x) \text{€}\ ax^p \text{€} \ = \ \text{€}\ (u \mid \nabla_x) ax^p \text{€} \tag{11}$$

$$= \begin{cases} 0 \text{ if } \ p = 0 \\ p \text{€}\ aux^{p-1} \text{€} \ \text{ if } \ p \neq 0. \end{cases}$$

Proof.- If $p \neq 0$ and $\varepsilon \in \mathbb{R}$

$$(u \mid \nabla_x) \text{€}\ ax^p \text{€} \ = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \text{€}\ a(x + \varepsilon u)^p \text{€} \ = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \text{€}\ a \sum_{k=0}^{p} \binom{p}{k} x^{p-k} \varepsilon^k u^k \text{€}$$

$$= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \sum_{k=0}^{p} \varepsilon^k \binom{p}{k} \text{€}\ ax^{p-k} u^k \text{€} \ = p \text{€}\ aux^{p-1} \text{€}\ .$$

Proposition 2.- *Let* $u \in S \oplus V$, $a \in \mathbb{R}_{0,n}$ $p \in \mathbb{Z} \setminus \{0\}$, *then*

$$(u \mid \nabla_x) \text{€}\ ax^p \text{€} \ = \ \text{€}\ (u \mid \nabla_x) ax^p \text{€} \ = p \text{€}\ aux^{p-1} \text{€}\ . \tag{12}$$

Proof.- We have only to work out the case $p < 0$. If $\|1 - x\| \ < 1$

$$(u \mid \nabla_x) \text{€}\ ax^{-p} \text{€} = (u \mid \nabla_x) \text{€}\ a\Big(1 - (1 - x^p)\Big)^{-1} \text{€}$$

$$= (u \mid \nabla_x) \text{€}\ a \sum_{q=0}^{\infty} (1 - x^p)^q \text{€} \ = \sum_{q=0}^{\infty} \text{€}\ (u \mid \nabla_x)\ a(1 - x^p)^q \text{€}$$

$$= \ \text{€}\ (u \mid \nabla_x)\ a \ \sum_{q=0}^{\infty} (1 - x^p)^q \text{€} \ = \ \text{€}\ (u \mid \nabla_x)\ ax^p \text{€} \ = p \text{€}\ aux^{p-1} \text{€}\ .$$

Theorem 4.- Let Ω be an open set of $S \oplus V$ with $0 \in \Omega$. Let $f : \ \Omega \to S \oplus V$ such that locally :

$$f(x) = \sum_{\alpha} \ P_\alpha(x) c_\alpha + \sum_{\mid \beta \mid < B} S_\beta(x) d_\beta \tag{13}$$

with $c_\alpha \in \mathbb{R}$ and $d_\beta \in \mathbb{R}$. Then for all $u \in V$ and $x \neq 0$ we have

$$\frac{\partial}{\partial x_0} \text{€}\ uf(x) \text{€} \ \ - (u \mid \nabla_x) \text{€}\ f(x) \text{€} \ = 0. \tag{14}$$

Remark.- We get exactly the classical Cauchy-Riemann equations. When $n = 1$, that is, in the $\mathbb{C}$ case, taking $u = i\lambda$, $\lambda \in \mathbb{R}$, we get these well-known equations. When n is odd, such function is holomorphic cliffordian and we say that it is with scalar coefficients.

Proof.- By uniform convergence, we have only to compare

$$\frac{\partial}{\partial x_0} \text{€}\ uP_\alpha(x)\ \text{€}\ = \frac{\partial}{\partial x_0} \text{€}\ uk_\alpha \partial^\alpha\ x^{2|\ \alpha\ |-1}\ \text{€}\ = k_\alpha\ \partial^\alpha(2\ |\alpha|\ -1)\ \text{€}\ ux^{2|\ \alpha\ |-2}\ \text{€}$$

$$(u \mid \nabla_x)\ \text{€}\ P_\alpha(x)\ \text{€}\ = \ \text{€}\ (u \mid \nabla_x)k_\alpha \partial^\alpha x^{2|\ \alpha\ |-1}\ \text{€}\ = k_\alpha \partial^\alpha(2\ |\alpha|\ -1)\ \text{€}\ u\ x^{2|\ \alpha\ |-2}\ \text{€}\ .$$

For the S_β, we have

$$\frac{\partial}{\partial x_0} \text{€}\ uS_\beta(x)\ \text{€}\ = \frac{\partial}{\partial x_0} \text{€}\ uh_\beta\ \partial^\beta x^{-1}\ \text{€}\ = -h_\beta\ \partial^\beta\ \text{€}\ ux^{-2}\ \text{€}$$

$$(u \mid \nabla_x)\text{€}S_\beta(x)\text{€} = \text{€}(u \mid \nabla_x)h_\beta\ \partial^\beta x^{-1}\text{€} = h_\beta\ \partial^\beta\text{€}(u \mid \nabla_x)x^{-1}\text{€} = -h_\beta\partial^\beta\text{€}ux^{-2}\text{€}.$$

Remark.- For this type of holomorphic Cliffordian function f and for $x \neq 0$,

$$\lim_{h\to 0}\ \text{€}\ \frac{f(x+h)-f(x)}{h}\ \text{€}\ ,$$

does not depend on the particular paravector h, because this is true for x^p, hence also for $P_\alpha(x)$, and $S_\beta(x)$, and therefore for f.

3.7 Taylor formula

Lemma 3.- Let $p \in \mathbb{Z}$, $q \in \mathbb{N}$, $u \in V$. Then

$$\frac{\partial^q}{\partial x_0^q} \text{€}\ u^q\ x^p\ \text{€}\ = (u \mid \nabla_x)^q\ x^p.$$

Proof.- iterate (11).

Using scalar derivations this implies

$$\frac{\partial^q}{\partial x_0^q} \text{€}\ u^q\ P_\alpha(x)\ \text{€}\ = (u \mid \nabla_x)^q\ \text{€}\ P_\alpha(x)\ \text{€}$$

$$\frac{\partial^q}{\partial x_0^q} \text{€}\ u^q\ S_\beta(x)\ \text{€}\ = (u \mid \nabla_x)^q\ \text{€}\ S_\beta(x)\ \text{€}.$$

If f is of the same type as in theorem 4 we have

$$\frac{\partial^q}{\partial x_0^q} \text{€}\ u^q\ f(x)\ \text{€}\ = (u \mid \nabla_x)^q\ \text{€}\ f(x)\ \text{€}. \tag{15}$$

Theorem 5 (Taylor series).- Let f be an holomorphic Cliffordian function with scalar coefficients, then we have :

$$\text{€}\ f(a+x)\ \text{€}\ = \sum_{k=0}^{\infty} \frac{1}{k!}\ \text{€}\ x^k\ \frac{\partial^k f}{\partial a_0^k}\ (a)\ \text{€}.$$

Proof.- Put $x = x_0 + \vec{x}$. Since f is real analytic, we have

$$\begin{aligned}
f(a+x) &= \sum_{k=0}^{\infty} \frac{1}{k!} \, (x \mid \nabla_a)^k \, f(a) \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \left(x_0 \, \frac{\partial}{\partial a_0} + (\vec{x} \mid \nabla_a) \right)^k \, f(a) \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{r+s=k} \binom{k}{r} \, x_0^r \, \frac{\partial^r}{\partial a_0^r} \, (\vec{x} \mid \nabla_a)^s \, f(a) \\
€ \; f(a+x) \; € &= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{r+s=k} \binom{k}{r} x_0^r \, \frac{\partial^r}{\partial a_0^r} \, \frac{\partial^s}{\partial a_0^s} \; € \; \vec{x}^s f(a) \; € \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{r+s=k} \binom{k}{r} \; € \; x_0^r \, \vec{x}^s \, \frac{\partial^k}{\partial a_0^k} \, f(a) \; € \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \; € \; x^k \, \frac{\partial^k}{\partial a_0^k} \, f(a) \; €.
\end{aligned}$$

3.8 Differential calculus

In this paragraph, n is odd.

Let ω be a differential form with values in $\mathbb{R}_{0,n}$. Then there exist scalar differential forms ω_I such that

$$\omega = \Sigma \, \omega_I \, e_I.$$

We define

$$€ \; \omega \; € \; := \Sigma \, \omega_I \; € \; e_I \; €$$

and then the exterior derivative

$$\begin{aligned}
d \; € \; \omega \; € \quad &= \; \Sigma \, d\omega_I \; € \; e_I \; € \\
&= \; \Sigma \; € \; d\omega_I \; e_I \; €
\end{aligned}$$

so that

$$d \; € \; \omega \; € = \; € \; d\omega \; €.$$

Let $\mathcal{P}_v$ be the vectorial plane generated by 1 and v, $v \in V$, $v^2 = -1$. For a holomorphic Cliffordian function of the same type as in the previous theorem and Ω_v an open set in $\mathcal{P}_v$ with regular boundary, we have a Cauchy-Morera theorem.

Theorem 6.-

$$\int_{\partial\Omega_v} € \; f(x) \; (dx_0 + v \; d(\vec{x} \mid v) \; € = \int_{\Omega_v} € \; v \frac{\partial f(x)}{\partial x_0} - (v \mid \nabla_x) f(x) \; € \; dx_0 \wedge d(\vec{x} \mid v) = 0.$$

Proof.- Stokes theorem gives :

$$\int_{\partial\Omega_v} € \; f(x) \Big(dx_0 + v d(\vec{x} \mid v) \Big) \; € = \int_{\Omega_v} d \; € \; f(x) \Big(dx_0 + v d(\vec{x} \mid v) \Big) \; € =$$

$$\int_{\Omega_v} € \; df(x) \wedge \Big(dx_0 + v d(\vec{x} \mid v) \Big) \; €$$

Then we get

$$\int_{\Omega_v} \in (v \mid \nabla) f(x)\ d(\vec{x} \mid v) \wedge dx_0 + \frac{\partial f(x)}{\partial x_0}\ dx_0 \wedge v d(\vec{x} \mid v) \in =$$
$$\int_{\Omega_v} \in v\ \frac{\partial f(x)}{\partial x_0} - (v \mid \nabla) f(x) \in dx_0 \wedge d(\vec{x} \mid v) = 0.$$

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