Quadratic sets of a 3-dimensional locally projective regular planar space

Roberta Di Gennaro Nicola Durante *

Abstract

In this paper quadratic sets of a 3-dimensional locally projective regular planar space $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ of order n are studied and classified. It is proved that if in $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ there is a non-degenerate quadratic set $\mathbf{H}$, then the planar space is either $\mathrm{PG}(3, n)$ or $\mathrm{AG}(3, n)$. Moreover in the first case $\mathbf{H}$ is either an ovoid or an hyperbolic quadric, in the latter case $\mathbf{H}$ is either a cylinder with base an oval or a pair of parallel planes.

1 Introduction

A *linear space* is a pair $(\mathcal{S}, \mathcal{L})$, where S is a non-empty set of *points* and $\mathcal{L}$ is a non-empty set of proper subsets of $\mathcal{S}$ called *lines*, such that through every pair of distinct points there is a unique line and every line has at least two points.

Let $(\mathcal{S}, \mathcal{L})$ be a finite linear space. For every point P of $\mathcal{S}$, the *degree* of P is the number $[P]$ of lines through P; for every line l, the *length* of l is its cardinality. The integer n defined by $n + 1 = \max\{[P] : P \in \mathcal{S}\}$ is the *order* of the linear space. A subset T of the point set $\mathcal{S}$ of a linear space $(\mathcal{S}, \mathcal{L})$ is a *subspace* if it contains the line through any two of its points.

A *planar space* is a triple $(\mathcal{S}, \mathcal{L}, \mathcal{P})$, where $(\mathcal{S}, \mathcal{L})$ is a linear space and $\mathcal{P}$ is a non-empty family of proper subspaces of $(\mathcal{S}, \mathcal{L})$, called *planes*, satisfying the following properties:

*Work done within the project "Strutture geometriche, Combinatoria e loro applicazioni" of M.U.R.S.T. Partially supported also from University "Federico II" of Naples.

Received by the editors October 2002.

Communicated by H. Van Maldeghem.

1991 *Mathematics Subject Classification :*

Key words and phrases :

(p_1) through any three non-collinear points there is a unique plane containing them;

(p_2) every plane contains at least three non-collinear points.

Let $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ be a finite planar space. In this paper v is the number of points, b is the number of lines and c is the number of planes of the planar space $(\mathcal{S}, \mathcal{L}, \mathcal{P})$.

For every plane π of $\mathcal{P}$, denote by $\mathcal{L}_\pi$ the set of the lines of $\mathcal{L}$ contained in π and by n_π the order of the linear space $(\pi, \mathcal{L}_\pi)$. The integer $n = \max\{n_\pi : \pi \in \mathcal{P}\}$ is the *order* of the planar space.

For any point X of $\mathcal{S}$, the *star of lines* with center X is the set of all lines through X.

Let π be a plane of $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ and let X be a point of π: the *pencil of lines* with center X in π is the set of all lines through X contained in π. If every pencil of lines has at least three lines we have a *thick* planar space. Two *skew* lines are two non-coplanar lines of a planar space. Two *parallel* lines are two lines ℓ and ℓ' such that either $\ell = \ell'$ or ℓ and ℓ' are coplanar and $\ell \cap \ell' = \emptyset$.

A planar space $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ is *embeddable* in a projective space $\mathbf{P}$ if there is an injection of $\mathcal{S}$ into the point set of $\mathbf{P}$ preserving collinearities and coplanarities as well as non-collinearities and non-coplanarities.

A planar space $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ is 3-*dimensional locally projective* if its proper subspaces are points, lines and planes and for every point P of $\mathcal{S}$, the linear space $(\mathcal{L}_P, \mathcal{P}_P)$ whose points are the lines through P and whose lines are the pencils of lines with center P, is a (non-degenerate) projective plane.

If $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ is a thick planar space of order n, it is easy to see that the property of being 3-dimensional locally projective is equivalent to the property that its planes pairwise intersect either in the empty set or in a line.

A finite planar space $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ is a *k-regular planar space* if all lines have the same length $k+1$. We will sometimes simply say that the planar space is *regular*.

Observe that, if $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ is a regular locally projective planar space of order n, then in every plane the pencils have $n+1$ lines and hence every plane has $(n+1)k+1$ points.

Throughout this paper $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ is a 3-dimensional locally projective regular planar space of order n.

It is not difficult to see that $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ satisfies the following properties.

(a) Through every point there are n^2+n+1 lines and n^2+n+1 planes.

(b) In every plane the pencils of lines have cardinality $n+1$.

(c) Through every line there are $n+1$ planes.

(d) In every plane there are $\dfrac{(nk+k+1)(n+1)}{k+1}$ lines.

(e) The number of lines is $b = \dfrac{((n^2+n+1)k+1)(n^2+n+1)}{k+1}$.

(f) The number of point is $(n^2+n+1)k+1 \leq n^3+n^2+n+1$.

The finite regular locally projective planar spaces have been studied by J. Doyen and X. Hubaut in [4]. They proved the following result.

Theorem 1.1. *Let $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ be a 3-dimensional regular locally projective planar space of order n, then three cases are possible.*

(C1) $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ *is isomorphic to* $\mathrm{PG}(3, n)$.

(C2) $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ *is isomorphic to* $\mathrm{AG}(3, n)$.

(C3) The order n of $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ satisfies $n = k + s$ with either $s = k^2 + k + 1$ or $s = (k + 1)^3 + 1$.

Remark. The spaces of type *(C3)* are called Lobachevsky spaces. If $s = k^2 + k+1$, then the smallest case $k = 1$ gives the unique $3-(22, 6, 1)$ design, i.e. the Witt design on 22 points. It has been proved in [6] that there cannot exist an example for $k = 2$. For $k \geq 3$ no example is known either. If $s = (k + 1)^3 + 1$, then the smallest case $k = 1$ would give a $3-(112, 12, 1)$ design that should be the extension of a projective plane of order 10. It is known that such a design cannot exists since there are no projective planes of order 10.

2 Quadratic sets

Let $\mathbf{K}$ be a set of points of $\mathcal{S}$. A line ℓ is *tangent* to $\mathbf{K}$ if either it is contained in $\mathbf{K}$ or it has exactly a point in common with $\mathbf{K}$. In the first case the line ℓ will be called a $\mathbf{K}$-*line*, in the latter case ℓ will be called a 1-*tangent* line to $\mathbf{K}$. A non-tangent line to $\mathbf{K}$ will be called *external* to $\mathbf{K}$ if it has empty intersection with $\mathbf{K}$, *secant* to $\mathbf{K}$ otherwise.

A plane π is *tangent* to $\mathbf{K}$ at a point P if each line through P in π is tangent to $\mathbf{K}$. A non-tangent plane to $\mathbf{K}$ will be called *external* to $\mathbf{K}$ if it has empty intersection with $\mathbf{K}$, *secant* to $\mathbf{K}$ otherwise.

For each point $P \in \mathbf{K}$ we can define the *tangent subset* $\mathbf{K}_P$ of $\mathbf{K}$ at P as the union of all tangent lines to $\mathbf{K}$ at P.

A *quadratic set* of $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ is a non-empty subset $\mathbf{H}$ of points of $\mathcal{S}$ such that each line that meets $\mathbf{H}$ in more than two points, is contained in $\mathbf{H}$, and such that for each point $P \in \mathbf{H}$ the tangent subset $\mathbf{H}_P$ is either a plane of $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ or the full point set $\mathcal{S}$.

A point P of $\mathbf{H}$ is *singular* if $\mathbf{H}_P = \mathcal{S}$. A quadratic set $\mathbf{H}$ of $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ is called *non-degenerate* if it has no singular points.

Let $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ be a non-degenerate planar space of order n and let π be a plane of $(\mathcal{S}, \mathcal{L}, \mathcal{P})$. A k-arc in π is a set Γ of k points of π no three collinear. It is easy to see that $k \leq n + 2$ and $k = n + 2$ if and only if n is even. Every $(n + 1)$-arc in π is called an *oval* of π.

A set C of points meeting every line in at most two points is a *cap* of $(\mathcal{S}, \mathcal{L}, \mathcal{P})$.

Planar spaces containing special types of caps have been studied by G. Tallini in [7]. He proved the following theorem.

Theorem 2.1. *Let $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ be a thick finite planar space with all lines of the same cardinality $k + 1$ and such that all planes have the same number of points. Let Ω be* *a cap of* $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ *such that for every point* P *of* Ω*, the union of the tangent lines at* P *is a subspace* τ_P *meeting every plane through* P*, but not in* τ_P*, in a line. Then* k *is a prime power,* $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ *is* $\mathrm{PG}(3, k)$ *and* Ω *is one of its ovoids.*

In the sequel we will also need the following result that follows immediately from a theorem of M. Hall Jr. [5].

Theorem 2.2. *Let* $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ *be a planar space of order* n *and suppose that every plane of* $\mathcal{P}$ *is an affine plane. If the parallelism between lines is transitive, then* n *is a prime power and* $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ *is* $\mathrm{AG}(d, n)$.

From this theorem follows easily the following lemma that will be useful for us.

Lemma 2.1. *Let* $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ *be a* 3*-dimensional locally projective planar space of order* n*. If every plane of* $\mathcal{P}$ *is an affine plane, then* $(\mathcal{S}, \mathcal{L}, \mathcal{P}) = \mathrm{AG}(3, n)$.

Proof. Let ℓ, ℓ', ℓ'' be three lines with $\ell \| \ell'$ and $\ell \| \ell''$. By the previous theorem, we only need to prove that $\ell' \| \ell''$. We also may assume that the lines ℓ, ℓ', ℓ'' do not lie in a common plane (hence they are also pairwise distinct lines), otherwise the theorem is proved. Suppose, by way of contradiction, that $\ell' \cap \ell'' \neq \emptyset$. Then the lines ℓ' and ℓ'' meet in a point P. It follows that $\pi = \langle \ell', \ell'' \rangle$ is a plane. Let $\pi' = \langle \ell', \ell \rangle$ and $\pi'' = \langle \ell'', \ell \rangle$, then $\pi' \cap \pi'' = \ell$. But $P \in \ell' \cap \ell''$ hence $P \in \pi' \cap \pi''$ and so $P \in \ell$, that is a contradiction since $\ell \| \ell'$ and $\ell \| \ell''$. Hence $\ell' \cap \ell'' = \emptyset$. It remains to prove that ℓ' and ℓ'' are coplanar. Let P' be a point on ℓ' and let $\pi^* = \langle P', \ell'' \rangle$, then $P' \in \pi^* \cap \pi'$. Hence $\pi^* \cap \pi'$ is a line through P' with no common points with ℓ. But the unique line through P' contained in π' and with no common points with ℓ is ℓ'. Hence $\pi' \cap \pi'' = \ell'$ and so ℓ' and ℓ'' are in a common plane. ■

Note that, if $|\ell| \geq 4$ for every line ℓ, the previous result also follows from the following theorem of F. Buekenhout [1].

Theorem 2.3. *Let* $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ *be a planar space of order* n *and suppose that every plane of* $\mathcal{P}$ *is an affine plane. If* $|\ell| \geq 4$ *for every line* ℓ*, then* n *is a prime power and* $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ *is* $\mathrm{AG}(d, n)$.

In $\mathrm{AG}(3, n)$ let π be a plane and Γ be an oval of π. A *cylinder* with base Γ is a set of $n + 1$ parallel lines each of them intersecting Γ in just one point.

In this paper we prove the following theorem.

Main Theorem. *Let* $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ *be a* 3*-dimensional locally projective regular planar space of order* n *and let* $\mathbf{H}$ *be a non-degenerate quadratic set of* $\mathcal{S}$*. Then the following cases are possible:*

(a) $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ *is* $\mathrm{PG}(3, n)$ *and* $\mathbf{H}$ *is either one of its ovoids or one of its hyperbolic quadrics;*

(b) $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ *is* $\mathrm{AG}(3, n)$ *and* $\mathbf{H}$ *is either the union of two disjoint planes or a cylinder with base an oval.*

Remarks.

1. This theorem shows that the definition of a quadratic set is good for projective spaces but not for the other 3-dimensional locally projective regular planar spaces as for instance non-singular quadrics in affine spaces do not survive here.

2. An unsuccessful attempt to give an axiomatic definition for non-degenerate quadrics in an affine space was given by F. Buekenhout in [2].

3 Proof of the main theorem

Throughout the rest of the paper $\mathbf{H}$ will be a non-degenerate quadratic set of $(\mathcal{S}, \mathcal{L}\,P)$. Moreover we can assume that in $\mathbf{H}$ there is an $\mathbf{H}$-line, otherwise we are in the hypothesis of Theorem 2.1 of Tallini and hence $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ is $\mathrm{PG}(3, n)$ and $\mathbf{H}$ is one of its ovoids.

We start with the following observation.

Observation 3.1. *Let P be a point of $\mathbf{H}$ and let h be the number of lines through P contained in $\mathbf{H}$. Then $|\mathbf{H}| = n^2 + hk + 1$ and hence h is independent on the point P.*

Lemma 3.1. *If there are two points P and Q in $\mathbf{H}$ such that $\mathbf{H}_P = \mathbf{H}_Q$, then $\mathbf{H}_P = \mathbf{H}_R$ for each $R \in \mathbf{H}_P \bigcap \mathbf{H}$. Moreover $\mathbf{H}_P \cap \mathbf{H}$ is either a line or a plane.*

Proof. Let P and Q be two points of $\mathbf{H}$ such that $\mathbf{H}_P = \mathbf{H}_Q$. If $\mathbf{H}_P \bigcap \mathbf{H}$ is the line joining P and Q, then for every point X on the line PQ holds $\mathbf{H}_X = \mathbf{H}_P$. Assume $\mathbf{H}_P \cap \mathbf{H} \neq PQ$ and let R be a point of $\mathbf{H}_P \bigcap \mathbf{H}$, not on the line PQ, then the lines RP and RQ are contained in $\mathbf{H}$ and hence $\mathbf{H}_R = \mathbf{H}_P$. For any point $Y \in PQ$, $Y \neq P, Y \neq Q$, there holds that $\mathbf{H}_Y = \mathbf{H}_P$ since YP and YQ are contained in $\mathbf{H}_Y$. We prove that in this case $\mathbf{H}_P \bigcap \mathbf{H}$ is a plane. Indeed let A, B be two points of $\mathbf{H}_P \bigcap \mathbf{H}$, then the line AB is contained in $\mathbf{H}$ since $\mathbf{H}_A = \mathbf{H}_B$ and hence $\mathbf{H}_P \bigcap \mathbf{H}$ is a proper subspace containing PQ and R and hence it is a plane. ∎

From Lemma 3.1, since $\mathbf{H}_P \bigcap \mathbf{H}$ is either a line or a plane of $\mathbf{H}$ and since h is the number of $\mathbf{H}$-lines through P, we have either $h = 1$ or $h = n + 1$. We first consider the case $h = 1$.

Proposition 3.1. *Let $\mathbf{H}$ be a quadratic set such that $\mathbf{H}_P = \mathbf{H}_Q$ for all points $Q \in \mathbf{H}_P \cap \mathbf{H}$ and let $h = 1$, then $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ is $\mathrm{AG}(3, n)$ and $\mathbf{H}$ is a cylinder with base an oval.*

Proof. Since through every point of $\mathbf{H}$ there are n 1-tangent lines and through each one of these lines there are n planes different from $\mathbf{H}_P$ that meet $\mathbf{H}$ in an $(n+1)$-arc, then counting in two ways the pairs (P, π) with $\mathbf{H} \bigcap \pi$ an $(n+1)$-arc we have:

$$n^2|\mathbf{H}| = (n+1)\alpha \tag{1}$$

where α is the number of planes that meet $\mathbf{H}$ in an $(n+1)$-arc. Counting $|\mathbf{H}|$ from lines through P we have $|\mathbf{H}| = n^2 + k + 1$ and so from Equation (1) it follows that $n+1$ divides $n^2(n^2+k+1) = (n^2-1+1)(n^2-1+k+2)$. Hence $n+1$ divides $k+2$, so $n \leq k+1$. Since $k \leq n$ we have either $k+1 = n$ or $k+1 = n+1$. If $k = n$ we have that all planes of the planar space are projective planes hence the planar space is $\mathrm{PG}(3,n)$ and we get a contradiction since in $\mathrm{PG}(3,n)$ non-degenerate quadratic sets are only the hyperbolic quadrics ([3]), in which case however $\mathbf{H}_P \neq \mathbf{H}_Q$ for any pair of distinct points $P, Q \in \mathbf{H}$.
Hence $k+1 = n$ and so all planes are affine planes. Moreover since $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ is a $3-$dimensional locally projective planar space it follows from Lemma 2.1 that $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ is $\mathrm{AG}(3,n)$. Since $h = 1$ counting the points of $\mathbf{H}$ we have $|\mathbf{H}| = n^2 + n$. Every secant plane π containing no $\mathbf{H}$-lines meets $\mathbf{H}$ in an oval. Let Ω be one of these ovals. Through every point of Ω there is a unique $\mathbf{H}$-line. Since $h = 1$, those $n+1$ lines do not have common points and so they cover n^2+n points of $\mathbf{H}$. Let $\ell_1, \ldots, \ell_{n+1}$ be those $n+1$ lines, we will prove that these lines are pairwise parallel. Let $P \in \ell_1$, since $\mathbf{H}_P \bigcap \mathbf{H} = \ell_1$, then the other n $\mathbf{H}$-lines through the points of Ω are parallel to $\mathbf{H}_P$. It follows that each one of these lines is contained in one of the $n-1$ planes parallel to $\mathbf{H}_P$ and different from $\mathbf{H}_P$ and so at least one of these planes π contains two of those lines ℓ_i and ℓ_j that are parallel. Let P' be a point of ℓ_j, $\mathbf{H}_{P'} \neq \pi$ since $H_{P'}$ contains only one $\mathbf{H}$-line ℓ_j. Then $\mathbf{H}_{P'}$ meets $\mathbf{H}_P$ in a line disjoint from $\mathbf{H}$ and so parallel to ℓ_1 and ℓ_j. It follows that ℓ_1 is parallel to ℓ_j so it is also parallel to ℓ_i. Using the same argument for all $\mathbf{H}$-lines through the points of Ω, since parallelism is transitive, it follows that the lines $\ell_1, \ldots, \ell_{n+1}$ are pairwise parallel and $\mathbf{H}$ is a cylinder with base Ω. ■

Next we study the case $h = n+1$.

Proposition 3.2. *Let* $\mathbf{H}$ *be a quadratic set of* $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ *such that* $\mathbf{H}_P = \mathbf{H}_Q$ *for every* $Q \in \mathbf{H}_P \cap \mathbf{H}$ *and let* $h = n+1$. *Then* n *is a prime power,* $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ *is* $\mathrm{AG}(3,n)$ *and* $\mathbf{H}$ *is a pair of parallel planes.*

Proof. We first prove that there is an external line ℓ to $\mathbf{H}$. Indeed let P be a point of $\mathbf{H}$ and let $\mathbf{H}_P$ be the tangent plane to $\mathbf{H}$ at P. Every plane π through P, different from $\mathbf{H}_P$, is a secant plane to $\mathbf{H}$ and $|\pi \bigcap \mathbf{H}| = n+k+1$. For every point $R \in \pi \setminus \mathbf{H}$ there are $(n+k+1)/2$ 2-secant lines contained in π. Since $(n+k+1)/2 < n+1$, there is at least one external line ℓ through R in π. All planes through ℓ are either external to $\mathbf{H}$ or meet $\mathbf{H}$ in $n+k+1$ points. Hence $|\mathbf{H}| = n^2 + (n+1)k + 1 = a(n+k+1)$, where a denotes the number of secant planes through ℓ. It follows that $n+k+1$ divides $n^2+(n+1)k+1$ and hence $n+k+1$ divides $2k+2$. So $n+k+1 \leq 2k+2$, hence $n \leq k+1$. So $k+1 = n$, hence all planes are affine planes and as above the planar space is $\mathrm{AG}(3,n)$. Since $|\mathbf{H}| = n^2 + (n+1)(n-1) + 1 = 2n^2$, the set $\mathbf{H}$ is the union of two parallel planes. ■

We can now assume that there exists a point P in $\mathbf{H}$ and a point $Q \in \mathbf{H}_P \bigcap \mathbf{H}$ such that $\mathbf{H}_P \neq \mathbf{H}_Q$. Then, from Lemma 3.1 it follows that $\mathbf{H}_R \neq \mathbf{H}_P$ for every $R \in \mathbf{H}_P \bigcap \mathbf{H}$. In this case we can prove the following proposition.

Proposition 3.3. *Let* $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ *be a* 3*-dimensional regular locally projective planar space of order* n*, let* $\mathbf{H}$ *be a non-degenerate quadratic set of* $\mathcal{S}$ *such that there exists* *a point P in $\mathbf{H}$ and a point Q in $\mathbf{H}_P \bigcap \mathbf{H}$ with $\mathbf{H}_P \neq \mathbf{H}_Q$. Then n is a prime power, $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ is* $\mathrm{PG}(3, n)$ *and $\mathbf{H}$ is one of its hyperbolic quadrics.*

Proof. We will prove that $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ is $\mathrm{PG}(3, n)$. Suppose by way of contradiction that $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ is not $\mathrm{PG}(3, n)$, then $k < n$. Let ℓ be the line through P and Q. The line ℓ is an $\mathbf{H}$-line since it is contained in $\mathbf{H}_P$. Moreover through ℓ there are at most $k + 1$ tangent planes (one for each point of ℓ) and since $k < n$, there is at least one secant plane π through ℓ. If we count the points of $\pi \bigcap \mathbf{H}$ from the lines through P in π we have $|\pi \bigcap \mathbf{H}| = n + k + 1$ since all lines through P, different from ℓ, are two-secant lines (otherwise π would be $\mathbf{H}_P$). Counting $|\pi \bigcap \mathbf{H}|$ from any point $P' \in \pi \bigcap \mathbf{H} \setminus \{P\}$ we see that through P' there is a unique line ℓ' contained in $\pi \bigcap \mathbf{H}$. Hence $\pi \bigcap \mathbf{H}$ is partitioned into $\mathbf{H}$-lines.
Furthermore in π there are no 1-tangent lines to $\mathbf{H}$ and so through each point $P' \in \pi \setminus \mathbf{H}$ (such a point exists since π is a secant plane) there are precisely $(n + k + 1)/2$ two-secant lines to $\mathbf{H}$. From this follows that $n + k + 1$ is even. Next consider the point Q in $\mathbf{H}_P$. Since $\mathbf{H}_P \neq \mathbf{H}_Q$, the unique tangent line through Q contained in $\mathbf{H}_P$ is the line ℓ, the other n lines through Q in $\mathbf{H}_P$ are all 2-secants lines. Hence $|\mathbf{H}_P \cap \mathbf{H}| = n + k + 1$. Let now T be a point of $\mathbf{H}_P$ not on $\mathbf{H}$, then the line TP is the unique tangent line through T contained in $\mathbf{H}_P$ and hence there are $(n + k)/2$ 2-secant lines through T in $\mathbf{H}_P$. It follows that $n + k$ is even, a contradiction. Hence $k = n$, so all planes are projective planes and hence $\mathcal{S}$ is $\mathrm{PG}(3, n)$. From [3] follows that the only quadratic set containing lines in a 3-dimensional projective space is the hyperbolic quadric. ∎

References

[1] F. Buekenhout, Ensembles quadratiques des espaces projectifs. Math. Z. 110, 306-318 (1969).

[2] F. Buekenhout, Une caractérisation des espaces affins basée sur la notion de droite. Math. Z. 111, 367-371 (1969).

[3] F. Buekenhout, On affine quadratic sets. Atti del Sem. Mat. Fis. Univ. Modena XXXV, 71-76 (1987).

[4] J. Doyen and X. Hubaut, Finite regular locally projective spaces. Math. Z. 119, 82-88 (1971).

[5] M. Hall Jr., Incidence axioms for affine geometry. J. of Algebra. 21, 535-547 (1972).

[6] C. Huybrechts, The non-existence of 3-dimensional locally projective spaces of orders $(2, 9)$, J. Combin. Theory Ser. A 71 no. 2, 340-342 (1995).

[7] G. Tallini, Ovoids and caps in planar spaces. Atti Conv. Combinatorics 1984, 347-353 North Holland (1986).

Dipartimento di Matematica e Applicazioni,
Università di Napoli "Federico II",
Complesso di Monte S. Angelo - via Cintia,
80126 Napoli, Italy.
e-mail: rdigenna@unina.it, ndurante@unina.it.