The Clifford-Laguerre Continuous Wavelet Transform

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Abstract

Higher dimensional wavelets are constructed in the framework of Clifford analysis by taking the Clifford-monogenic extension of specific functions. Clifford-monogenic functions are direct higher dimensional generalizations of holomorphic functions in the complex plane. In this way also new generalized polynomials, the so-called Clifford-Laguerre polynomials, are obtained.

1 Introduction

The wavelet transform has become quite a standard tool in numerous research and application domains and its popularity has increased rapidly over the last few decades.
The main idea in wavelet theory is to analyse a signal according to scale. Wavelets are functions that oscillate like a wave in a limited portion of time or space and vanish outside of it, i.e. they are wave-like but localized functions. One chooses a particular wavelet, dilates or contracts it (to meet a given scale) and shifts it, while looking into its correlations with the analyzed signal. The signal correlations with wavelets dilated to large scales reveal gross (”rude”) features, while at small scales fine signal structures are discovered.
In such a scanning through a signal, the scale and the position can vary continuously or in discrete steps. The former gives rise to the continuous wavelet transform (abbreviated CWT), the latter to the discrete wavelet transform (abbreviated DWT).

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The CWT and DWT enjoy more or less opposite properties and both have their specific field of application. The CWT is a successful tool for the analysis of signals and feature detection in signals, while the DWT provides a powerful technique for e.g. data compression and signal reconstruction. This paper deals with the CWT. For a standard introduction to wavelet analysis we refer the reader to [7].
Wavelets constitute a family of functions derived from one single function called the mother wavelet. The mother wavelet $\psi(x)$ generates the other wavelets of the family by change of scale a (i.e. by dilation) and by change of position b (i.e. by translation):

$$\psi_{a,b}(x) = \frac{1}{\sqrt{a}}\psi\left(\frac{x-b}{a}\right), \quad a>0,\ b\in\mathbb{R}.$$

In the theory some conditions on the mother wavelet ψ are needed. We request ψ to be an L_2-function (finite energy) which is well localized both in the time domain and in the frequency domain. Moreover it has to satisfy the so-called admissibility condition:

$$C_\psi := \int_{-\infty}^{+\infty} \frac{|\widehat{\psi}(u)|^2}{|u|} du < +\infty,$$

where $\widehat{\psi}$ denotes the Fourier transform of ψ. In the case where ψ is also in L_1, this admissibility condition implies

$$\int_{-\infty}^{+\infty} \psi(x)dx = 0,$$

hence ψ must be oscillating.
In practice, applications impose additional requirements, among which a given number of vanishing moments:

$$\int_{-\infty}^{+\infty} x^n\psi(x)dx = 0, \quad n = 0,1,\ldots,N.$$

This means that the corresponding CWT:

$$\begin{aligned} F(a,b) &= <\psi_{a,b}, f> \\ &= \frac{1}{\sqrt{a}}\int_{-\infty}^{+\infty} \overline{\psi\left(\frac{x-b}{a}\right)} f(x)dx \end{aligned}$$

will filter out polynomial behaviour of the signal up to degree N, making it adequate at detecting singularities.
When considering two L_2-functions f and g with CWT respectively F and G, the following inner product may be introduced:

$$[F,G] = \frac{1}{C_\psi}\int_{-\infty}^{+\infty}\int_0^{+\infty} \overline{F}(a,b)G(a,b)\frac{da}{a^2}db.$$

Taking into account the above mentioned admissibility condition for the mother wavelet ψ, the corresponding Parseval formula is readily obtained:

$$[F,G] = <f,g> .$$

In other words, as a consequence of the admissibility condition the CWT is an isometry from the space of signals into the space of transforms.
This implies the existence of the inverse transform:

$$f(x) = \frac{1}{C_\psi} \int_{-\infty}^{+\infty} \int_0^{+\infty} F(a,b)\psi_{a,b}(x) \frac{da}{a^2} db,$$

which means that the signal $f(x)$ may be reconstructed exactly from its transform $F(a,b)$. In other words, the CWT provides a decomposition of the signal in terms of the analyzing wavelets $\psi_{a,b}$ with coefficients $F(a,b)$.
The CWT may be extended to higher dimension while still enjoying the same properties as in the one dimensional case. Many wavelets are available for practical applications, often linked to a specific problem. Special attention should be paid to the two dimensional CWT

$$F(a,\underline{b},\theta) = \frac{1}{a} \int_{\mathbb{R}^2} \overline{\psi\left(\frac{r_{-\theta}(\underline{x}-\underline{b})}{a}\right)} f(\underline{x}) d\underline{x}$$

where now the mother wavelet ψ is not only translated by $\underline{b} \in \mathbb{R}^2$ and dilated by $a > 0$, but also rotated by an angle $\theta \in [0, 2\pi[$ (see [1]). It is an efficient tool for detecting oriented features of the signal provided the mother wavelet ψ contains itself an intrinsic orientation.
The aim of this paper is to construct higher dimensional wavelets in the framework of Clifford analysis. Clifford analysis deals with so-called monogenic functions; these are direct higher dimensional generalizations of holomorphic functions in the complex plane (see section 2).
Starting point is a specific real-analytic function in an open region of $\mathbb{R}^m$, which is monogenically extended to an appropriate domain in $\mathbb{R}^{m+1}$. Expressing the monogenicity of this extension leads to a recurrence relation for new special functions, most of which may be used as mother wavelets.
This technique was already successfully applied for constructing wavelets on the basis of the Clifford generalizations of the Hermite polynomials and the Gegenbauer polynomials (see [4], [5] and [3]).
In this paper we first construct the so-called Clifford-Laguerre polynomials, a certain generalization to Clifford analysis of the traditional Laguerre polynomials on the real line (section 3). After establishing an orthogonality relation for these Clifford-Laguerre polynomials, we select some of them to be candidates for mother wavelets (section 4). The corresponding CWT follows readily (section 5).

2 Clifford analysis

Clifford analysis (see e.g. [2] and [8]) offers a function theory which is a higher dimensional analogue of the holomorphic functions of one complex variable.
Consider functions defined in $\mathbb{R}^m$ $(m > 1)$ and taking values in the Clifford algebra $\mathbb{R}_m$ or its complexification $\mathbb{C}_m$. If $(e_1, \ldots, e_m)$ is an orthonormal basis of $\mathbb{R}^m$, the non-commutative multiplication in the Clifford algebra is governed by the rule:

$$e_j e_k + e_k e_j = -2\delta_{j,k}, \quad j,k = 1,\ldots,m.$$

Two anti-involutions on the Clifford algebra are important.
Conjugation is defined as the anti-involution for which

$$\overline{e_j} = -e_j, \quad j = 1, \ldots, m$$

with the additional rule

$$\bar{i} = -i$$

in the case of $\mathbb{C}_m$.
Inversion is defined as the anti-involution for which

$$e_j^\dagger = e_j, \quad j = 1, \ldots, m.$$

In what follows $\mathbb{R}_m^k$ denotes the subspace of k-vectors, i.e. the space spanned by the products of k different basis vectors.
The Euclidean space $\mathbb{R}^m$ is embedded in the Clifford algebras $\mathbb{R}_m$ and $\mathbb{C}_m$ by identifying $(x_1, \ldots, x_m)$ with the vector variable $\underline{x}$ given by

$$\underline{x} = \sum_{j=1}^{m} e_j x_j,$$

whereas the Euclidean space $\mathbb{R}^{m+1}$ is identified with $\mathbb{R}_m^0 \oplus \mathbb{R}_m^1$ by considering $(x_0, x_1, \ldots, x_m)$ as $x_0 + \underline{x}$.
The product of two vectors splits up into a scalar part and a so-called bivector part:

$$\underline{x}\underline{y} = - < \underline{x}, \underline{y} > + \underline{x} \wedge \underline{y}$$

where

$$< \underline{x}, \underline{y} >= \sum_{j=1}^{m} x_j y_j$$

and

$$\underline{x} \wedge \underline{y} = \sum_{j=1}^{m} \sum_{k=j+1}^{m} e_j e_k (x_j y_k - x_k y_j).$$

In particular

$$\underline{x}^2 = - < \underline{x}, \underline{x} >= -|\underline{x}|^2.$$

The even subalgebra $\mathbb{R}_m^+$ of the Clifford algebra $\mathbb{R}_m$ is defined by

$$\mathbb{R}_m^+ = \sum_{k \text{ even}} \oplus \mathbb{R}_m^k.$$

The Clifford group $\Gamma(m)$ of the Clifford algebra $\mathbb{R}_m$, consists of those invertible elements λ in $\mathbb{R}_m$ for which the action $\lambda \underline{x} \overline{\lambda}$ on a vector $\underline{x} \in \mathbb{R}_m^1$ is again a vector. Its subgroup Γ^+ is the intersection of Γ with the even subalgebra $\mathbb{R}_m^+$. The Spin-group $Spin(m)$ is the subgroup of Γ^+ of those elements $s \in \Gamma^+$ for which $ss^\dagger = 1$, or equivalently

$$Spin(m) = \{s = \underline{\omega}_1 \ldots \underline{\omega}_{2l};\ \underline{\omega}_j \in S^{m-1}, j = 1, \ldots, 2l, l \in \mathbb{N}\},$$

where S^{m-1} denotes the unit sphere in $\mathbb{R}^m$. The Spin-group is a two-fold covering group of the rotation group $SO(m)$. For $T \in SO(m)$, there exists $s \in Spin(m)$ such

that $T(\underline{x}) = s\underline{x}\overline{s}$, $\underline{x} \in \mathbb{R}^m$.
Two representations of $Spin(m)$ on the space of Clifford valued functions in $L_2(\mathbb{R}^m)$, are the L-representation

$$L(s)f(\underline{x}) = sf(\overline{s}\underline{x}s), \quad s \in Spin(m)$$

and the H-representation

$$H(s)f(\underline{x}) = sf(\overline{s}\underline{x}s)\overline{s}, \quad s \in Spin(m).$$

An $\mathbb{R}_m$- or $\mathbb{C}_m$-valued function $F(x_1, \ldots, x_m)$, respectively $F(x_0, x_1, \ldots, x_m)$ is called left-monogenic in an open region of $\mathbb{R}^m$, respectively $\mathbb{R}^{m+1}$, if in that region:

$$\partial_{\underline{x}} F = 0, \quad \text{respectively} \quad (\partial_{x_0} + \partial_{\underline{x}})F = 0.$$

Here $\partial_{\underline{x}}$ is the Dirac operator

$$\partial_{\underline{x}} = \sum_{j=1}^{m} e_j \partial_{x_j},$$

which splits the Laplacian in $\mathbb{R}^m$:

$$\Delta_m = -\partial_{\underline{x}}^2,$$

whereas $\partial_{x_0} + \partial_{\underline{x}}$ is the Cauchy-Riemann operator for which

$$\Delta_{m+1} = (\partial_{x_0} + \partial_{\underline{x}})(\partial_{x_0} + \overline{\partial_{\underline{x}}}).$$

The notion of right-monogenicity is defined in a similar way, letting act the Dirac operator or the Cauchy-Riemann operator from the right.
We denote by $\mathcal{H}(k)$ the set of harmonic homogeneous polynomials S_k of degree k in $\mathbb{R}^m$:

$$\Delta_m S_k(\underline{x}) = 0 \quad \text{and} \quad S_k(t\underline{x}) = t^k S_k(\underline{x}),$$

usually called solid spherical harmonics.
If $\underline{\Omega} \subset \mathbb{R}^m$ is open, then an open neighbourhood Ω of $\underline{\Omega}$ in $\mathbb{R}^{m+1}$ is said to be x_0-normal if for each $x \in \Omega$ the line segment $\{x + t; t \in \mathbb{R}\} \cap \Omega$ is connected and contains exactly one point in $\underline{\Omega}$.
Considering $\mathbb{R}^m$ as the hyperplane $x_0 = 0$ in $\mathbb{R}^{m+1}$, a real-analytic function $f(\underline{x})$ in an open connected domain $\underline{\Omega}$ in $\mathbb{R}^m$ can be uniquely extended to a monogenic function $F(x_0, \underline{x})$ in an open connected and x_0-normal neighbourhood Ω of $\underline{\Omega}$ in $\mathbb{R}^{m+1}$. This so-called Cauchy-Kowalewskaia (CK-) extension of $f(\underline{x})$ is given by

$$F(x_0, \underline{x}) = \sum_{k=0}^{\infty} (-1)^k \frac{x_0^k}{k!} \partial_{\underline{x}}^k f(\underline{x}).$$

The CK-extension procedure leads to the CK-product which, despite the non-commutativity of the Clifford algebra, preserves the monogenicity of the factors: the CK-product of two monogenic functions in $\mathbb{R}^{m+1}$ is the CK-extension to $\mathbb{R}^{m+1}$ of the product of the real-analytic restrictions to $\mathbb{R}^m$. For example, if $z_j,\ j = 1, \ldots, m$ denote the monogenic variables in $\mathbb{R}^{m+1}$:

$$z_j = x_j - x_0 e_j,$$

then their CK-product is the CK-extension of $x_j x_k$, given by

$$z_j \odot z_k = \frac{1}{2}(z_j z_k + z_k z_j).$$

In the sequel, the so-called Clifford-Heaviside functions

$$P^+ = \frac{1}{2}\left(1 + i\frac{\underline{x}}{|\underline{x}|}\right), \quad P^- = \frac{1}{2}\left(1 - i\frac{\underline{x}}{|\underline{x}|}\right)$$

play an important rôle; they were introduced independently by Sommen in [13] and McIntosh in [12].
These Clifford-Heaviside functions satisfy the relations

$$P^+ + P^- = 1; \quad P^+P^- = P^-P^+ = 0; \quad (P^+)^2 = P^+; \quad (P^-)^2 = P^-.$$

Throughout this paper the Fourier transform of $f(\underline{x})$ will be denoted by $\mathcal{F}(f(\underline{x}))(\underline{y})$:

$$\mathcal{F}(f(\underline{x}))(\underline{y}) = \int_{\mathbb{R}^m} \exp\left(-i < \underline{x}, \underline{y} >\right) f(\underline{x})\, dV(\underline{x}),$$

where $dV(\underline{x})$ stands for the Lebesgue measure on $\mathbb{R}^m$.

3 The Clifford-Laguerre polynomials

On the real line the generalized Laguerre polynomials $L_n^{(\alpha)}(x)$, for $\alpha > -1$, are defined by

$$L_n^{(\alpha)}(x) = x^{-\alpha}\frac{\exp(x)}{n!}\frac{d^n}{dx^n}\left(\exp(-x)x^{n+\alpha}\right), \quad n = 0, 1, 2, \ldots$$

They constitute an orthogonal basis for $L_2\left([0,\infty[, x^\alpha \exp(-x)\right)$ and satisfy the orthogonality relation

$$\int_0^\infty L_m^{(\alpha)}(x) L_n^{(\alpha)}(x) x^\alpha \exp(-x) dx = \Gamma(1+\alpha)\binom{n+\alpha}{n}\delta_{m,n}.$$

Starting point for the construction of a generalization to Clifford analysis of these classical Laguerre polynomials are the functions

$$F(\underline{x}) = \exp(-|\underline{x}|)|\underline{x}|^\alpha P^+ \quad \text{and} \quad G(\underline{x}) = \exp(-|\underline{x}|)|\underline{x}|^\alpha P^-; \quad \alpha \in \mathbb{R},$$

which are real-analytic in the open connected domain $\mathbb{R}^m \setminus \{\underline{0}\}$ in $\mathbb{R}^m$.
Consequently their CK-extensions, which can be written as

$$F^*(x_0, \underline{x}) = \exp(-|\underline{x}|) \sum_{k=0}^\infty \frac{x_0^k}{k!} |\underline{x}|^{\alpha-2k}\left(L^{+,+}_{k,m,\alpha}(\underline{x})P^+ + L^{+,-}_{k,m,\alpha}(\underline{x})P^-\right)$$

and analogously

$$G^*(x_0, \underline{x}) = \exp(-|\underline{x}|) \sum_{k=0}^\infty \frac{x_0^k}{k!} |\underline{x}|^{\alpha-2k}\left(L^{-,+}_{k,m,\alpha}(\underline{x})P^+ + L^{-,-}_{k,m,\alpha}(\underline{x})P^-\right),$$

exist in an open connected and x_0-normal neighbourhood Ω of $\mathbb{R}^m \setminus \{\underline{0}\}$ in $\mathbb{R}^{m+1}$. By definition F^* satisfies in Ω:

$$F^*(0,\underline{x}) = \exp(-|\underline{x}|)|\underline{x}|^\alpha P^+ \tag{1}$$

and

$$(\partial_{x_0} + \partial_{\underline{x}})F^*(x_0,\underline{x}) = 0. \tag{2}$$

From the monogenicity relation (2) we derive the recurrence relation

$$\begin{aligned} & L^{+,+}_{k+1,m,\alpha}(\underline{x})P^+ + L^{+,-}_{k+1,m,\alpha}(\underline{x})P^- \\ = \ & |\underline{x}|\underline{x}\Big(L^{+,+}_{k,m,\alpha}(\underline{x})P^+ + L^{+,-}_{k,m,\alpha}(\underline{x})P^-\Big) - (\alpha-2k)\underline{x} \\ & \Big(L^{+,+}_{k,m,\alpha}(\underline{x})P^+ + L^{+,-}_{k,m,\alpha}(\underline{x})P^-\Big) + \underline{x}^2\partial_{\underline{x}}\Big(L^{+,+}_{k,m,\alpha}(\underline{x})P^+ + L^{+,-}_{k,m,\alpha}(\underline{x})P^-\Big). \end{aligned}$$

As it follows from (1) that

$$L^{+,+}_{0,m,\alpha}(\underline{x}) = 1 \quad \text{and} \quad L^{+,-}_{0,m,\alpha}(\underline{x}) = 0,$$

we thus obtain consecutively:

$$\begin{aligned} L^{+,+}_{1,m,\alpha}(\underline{x}) &= i\underline{x}^2 + \left(\frac{1-m}{2} - \alpha\right)\underline{x} \\ L^{+,-}_{1,m,\alpha}(\underline{x}) &= \left(\frac{m-1}{2}\right)\underline{x} \\ L^{+,+}_{2,m,\alpha}(\underline{x}) &= -\underline{x}^4 + i(-2\alpha - m + 1)\underline{x}^3 + \left(\alpha(\alpha-2) + \frac{1}{2} - \frac{m}{2} + m\alpha\right)\underline{x}^2 \\ L^{+,-}_{2,m,\alpha}(\underline{x}) &= \left(\frac{m-1}{2}\right)\underline{x}^2 \\ L^{+,+}_{3,m,\alpha}(\underline{x}) &= -i\underline{x}^6 + \left(3\alpha + 3\frac{m}{2} - \frac{3}{2}\right)\underline{x}^5 + i\Big(\alpha(\alpha-2) + 3m\alpha - \frac{5}{2}m + 2 \\ &\quad +2\alpha^2 - 4\alpha + \frac{m^2}{2}\Big)\underline{x}^4 + \Big(-\alpha(\alpha-2)(\alpha-4) - \frac{3}{2}m\alpha^2 + 4m\alpha \\ &\quad +\frac{5}{2}\alpha - 2m + \frac{3}{2} - \frac{3}{2}\alpha^2 - \frac{m^2\alpha}{2} + \frac{m^2}{2}\Big)\underline{x}^3 \\ L^{+,-}_{3,m,\alpha}(\underline{x}) &= \left(\frac{1-m}{2}\right)\underline{x}^5 + i\left(-\frac{3}{2}m + 1 - \alpha + m\alpha + \frac{m^2}{2}\right)\underline{x}^4 \\ &\quad + \left(-2m\alpha + 2m + \frac{3}{2}\alpha - \frac{\alpha^2}{2} - \frac{3}{2} + \frac{m\alpha^2}{2} + \frac{m^2\alpha}{2} - \frac{m^2}{2}\right)\underline{x}^3 \end{aligned}$$

and so on.
Note that $L^{+,+}_{k,m,\alpha}(\underline{x})$ is a polynomial of degree $2k$ in $\underline{x}$, while $L^{+,-}_{k,m,\alpha}(\underline{x})$ is a polynomial of alternative degree $2k-1$ and $2k-2$ in $\underline{x}$.
Furthermore, the Clifford-Laguerre polynomials $L^{+,+}_{k,m,\alpha}(\underline{x})$ and $L^{+,-}_{k,m,\alpha}(\underline{x})$ satisfy the relation

$$\begin{aligned} & L^{+,+}_{k,m,\alpha}(\underline{x})P^+ + L^{+,-}_{k,m,\alpha}(\underline{x})P^- \\ = \ & (-1)^k \exp(|\underline{x}|)|\underline{x}|^{2k-\alpha}\partial^k_{\underline{x}}\Big(\exp(-|\underline{x}|)|\underline{x}|^\alpha P^+\Big). \end{aligned} \tag{3}$$

Naturally, similar formulae hold for the Clifford-Laguerre polynomials generated by the CK-extension G^*. It turns out that $L^{-,+}_{k,m,\alpha}(\underline{x})$, respectively $L^{-,-}_{k,m,\alpha}(\underline{x})$ is the complex conjugate of $L^{+,-}_{k,m,\alpha}(\underline{x})$, respectively $L^{+,+}_{k,m,\alpha}(\underline{x})$.

By means of formula (3) we obtain the following proposition.

Proposition
When $\alpha > -m$ and $2k < l$ we have the orthogonality relation

$$\int_{\mathbb{R}^m} \overline{L^{+,+}_{k,m,\alpha+2k}(\underline{x})}\left(L^{+,+}_{l,m,\alpha+2l}(\underline{x})P^+ + L^{+,-}_{l,m,\alpha+2l}(\underline{x})P^-\right)|\underline{x}|^\alpha \exp\left(-|\underline{x}|\right)dV(\underline{x}) = 0.$$

Proof:
As $L^{+,+}_{k,m,\alpha+2k}(\underline{x})$ is a polynomial of degree $2k$ in $\underline{x}$, it is sufficient to show that for each $t < l$:

$$\int_{\mathbb{R}^m} \underline{x}^t \left(L^{+,+}_{l,m,\alpha+2l}(\underline{x})P^+ + L^{+,-}_{l,m,\alpha+2l}(\underline{x})P^-\right)|\underline{x}|^\alpha \exp\left(-|\underline{x}|\right)dV(\underline{x}) = 0. \tag{4}$$

Introducing spherical co-ordinates in $\mathbb{R}^m$:

$$\underline{x} = r\underline{\omega}, \quad r = |\underline{x}|, \quad \underline{\omega} \in S^{m-1},$$

the function $L^{+,+}_{l,m,\alpha+2l}(\underline{x})P^+ + L^{+,-}_{l,m,\alpha+2l}(\underline{x})P^-$ takes the "axial" form

$$A_{2l}(r) + B_{2l}(r)\underline{\omega},$$

where A_{2l} and B_{2l} are polynomials with complex coefficients of degree $2l$ in the variable r.
Hence, if $t = 2s$ is even, the above integral becomes

$$(-1)^s \int_0^{+\infty} \int_{S^{m-1}} r^{2s}\left(A_{2l}(r) + B_{2l}(r)\underline{\omega}\right) r^\alpha \exp\left(-r\right) r^{m-1} dr dS(\underline{\omega}),$$

where $dS(\underline{\omega})$ stands for the Lebesgue measure on the unit sphere S^{m-1} in $\mathbb{R}^m$.
As $\underline{\omega}$ is a specific spherical harmonic:

$$\int_{S^{m-1}} \underline{\omega} dS(\underline{\omega}) = 0,$$

so that this integral can be further simplified to

$$A_m(-1)^s \int_0^{+\infty} A_{2l}(r) r^{2s+\alpha+m-1} \exp\left(-r\right)dr,$$

with

$$A_{m-1} = \frac{2\pi^{m/2}}{\Gamma\left(\frac{m}{2}\right)}$$

the area of the unit sphere S^{m-1}.
The above integral converges if $\alpha > -m - 2s$.
If $t = 2s+1$ is odd, we obtain in an analogous manner that the integral (4) converges

if $\alpha > -m - (2s+1)$.
Consequently, we conclude that the integral (4) converges for each $0 \leq t < l$ under the assumption $\alpha > -m$.
Using respectively relation (3) and Stokes's theorem, we obtain consecutively

$$\begin{aligned}
&\int_{\mathbb{R}^m} \underline{x}^t \left(L^{+,+}_{l,m,\alpha+2l}(\underline{x})P^+ + L^{+,-}_{l,m,\alpha+2l}(\underline{x})P^- \right) |\underline{x}|^\alpha \exp(-|\underline{x}|) dV(\underline{x}) \\
= \;& (-1)^l \int_{\mathbb{R}^m} \underline{x}^t \partial^l_{\underline{x}} \left(\exp(-|\underline{x}|)|\underline{x}|^{\alpha+2l} P^+ \right) dV(\underline{x}) \\
= \;& (-1)^l \Big\{ \int_{\partial\mathbb{R}^m} \underline{x}^t d\underline{\sigma} \partial^{l-1}_{\underline{x}} \left(\exp(-|\underline{x}|)|\underline{x}|^{\alpha+2l} P^+ \right) \\
& - \int_{\mathbb{R}^m} (\underline{x}^t \partial_{\underline{x}}) \partial^{l-1}_{\underline{x}} \left(\exp(-|\underline{x}|)|\underline{x}|^{\alpha+2l} P^+ \right) dV(\underline{x}) \Big\} \\
= \;& (-1)^{l+1} \int_{\mathbb{R}^m} (\underline{x}^t \partial_{\underline{x}}) \partial^{l-1}_{\underline{x}} \left(\exp(-|\underline{x}|)|\underline{x}|^{\alpha+2l} P^+ \right) dV(\underline{x}).
\end{aligned}$$

As $\underline{x}^t \partial_{\underline{x}} \approx \underline{x}^{t-1}$ and $t < l$, repeating this argument leads to the desired result. ■

4 The Clifford-Laguerre wavelets

For $\alpha > -m$ and $0 < l$, the above proposition implies that the L_1-functions

$$\begin{aligned}
\psi_{l,m,\alpha}(\underline{x}) &= \left(L^{+,+}_{l,m,\alpha+2l}(\underline{x})P^+ + L^{+,-}_{l,m,\alpha+2l}(\underline{x})P^- \right) |\underline{x}|^\alpha \exp(-|\underline{x}|) \\
&= (-1)^l \partial^l_{\underline{x}} \left(\exp(-|\underline{x}|)|\underline{x}|^{\alpha+2l} P^+ \right)
\end{aligned}$$

have zero momentum.
In order that $\psi_{l,m,\alpha}$ is also an L_2-function, we have to make the restriction $\alpha > -m/2$. Hence, for $\alpha > -m/2$ and $0 < l$, the functions $\psi_{l,m,\alpha}$ are good candidates for mother wavelets in $\mathbb{R}^m$, if at least they satisfy an appropriate admissibility condition (see section 5). We call them the Clifford-Laguerre wavelets.
The wavelets $\psi_{l,m,\alpha}$ have vanishing moments up to order $l-1$:

$$\int_{\mathbb{R}^m} \underline{x}^j \psi_{l,m,\alpha}(\underline{x}) dV(\underline{x}) = 0 \quad , j = 0, \ldots, l-1.$$

Their Fourier transform is

$$\mathcal{F}(\psi_{l,m,\alpha}(\underline{x}))(\underline{u}) = (-i)^l \underline{u}^l \mathcal{F}\left(\exp(-|\underline{x}|)|\underline{x}|^{\alpha+2l} P^+ \right)(\underline{u}),$$

where by definition

$$\begin{aligned}
&\mathcal{F}\left(\exp(-|\underline{x}|)|\underline{x}|^{\alpha+2l} P^+ \right)(\underline{u}) \\
= &\int_{\mathbb{R}^m} \exp(-i < \underline{x}, \underline{u} >) \exp(-|\underline{x}|)|\underline{x}|^{\alpha+2l} \frac{1}{2}\left(1 + i\frac{\underline{x}}{|\underline{x}|}\right) dV(\underline{x}). \qquad (5)
\end{aligned}$$

Introducing spherical co-ordinates

$$\underline{x} = r\underline{\omega}, \quad \underline{u} = \rho\underline{\xi}; \quad r = |\underline{x}|, \quad \rho = |\underline{u}|, \quad \underline{\omega} \in S^{m-1}, \quad \underline{\xi} \in S^{m-1}$$

expression (5) becomes

$$\begin{aligned} &\mathcal{F}\Big(\exp(-|\underline{x}|)|\underline{x}|^{\alpha+2l}P^+\Big)(\underline{u}) \\ = \;& \frac{1}{2}\int_0^{+\infty} \exp(-r) r^{\alpha+2l+m-1} dr \int_{S^{m-1}} \exp(-ir\rho < \underline{\omega}, \underline{\xi} >) dS(\underline{\omega}) \\ + \;& \frac{i}{2}\int_0^{+\infty} \exp(-r) r^{\alpha+2l+m-1} dr \int_{S^{m-1}} \exp(-ir\rho < \underline{\omega}, \underline{\xi} >)\underline{\omega} dS(\underline{\omega}). \end{aligned} \tag{6}$$

From the theory of the Fourier transform of radial functions and the theory of Bessel functions, it is well known that

$$\int_{S^{m-1}} \exp(-ir\rho < \underline{\omega}, \underline{\xi} >) dS(\underline{\omega}) = \frac{(2\pi)^{m/2} J_{m/2-1}(r\rho)}{r^{m/2-1}\rho^{m/2-1}},$$

with $J_{m/2-1}$ the Bessel function of the first kind of order $m/2-1$ (see [15]).
By applying this result, the first term of the right hand side of equation (6) becomes

$$\begin{aligned} &\frac{1}{2}\int_0^{+\infty} \exp(-r) r^{\alpha+2l+m-1} dr \int_{S^{m-1}} \exp(-ir\rho < \underline{\omega}, \underline{\xi} >) dS(\underline{\omega}) \\ = \;& \frac{(2\pi)^{m/2}}{2} \frac{1}{\rho^{m/2-1}} \int_0^{+\infty} \exp(-r) r^{\alpha+2l+m/2} J_{m/2-1}(r\rho) dr. \end{aligned}$$

By the assumption $\alpha > -m$, this can be further simplified to (see for e.g. [9])

$$\begin{aligned} &\frac{1}{2}\int_0^{+\infty} \exp(-r) r^{\alpha+2l+m-1} dr \int_{S^{m-1}} \exp(-ir\rho < \underline{\omega}, \underline{\xi} >) dS(\underline{\omega}) \\ = \;& \frac{(2\pi)^{m/2}}{2}\Gamma(\alpha+2l+m)\frac{\rho^{1-m/2}}{(1+\rho^2)^{(\alpha+2l+m/2+1)/2}} P^{1-m/2}_{\alpha+2l+m/2}\Big((1+\rho^2)^{-1/2}\Big), \end{aligned}$$

with

$$P^\mu_\nu(x) = \frac{1}{\Gamma(1-\mu)}\left(\frac{1+x}{1-x}\right)^{\mu/2} F\left(-\nu, \nu+1; 1-\mu; \frac{1-x}{2}\right); \quad -1 < x < 1$$

the associated Legendre function of the first kind and

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!}; \quad |z| < 1$$

the hypergeometric function.
To calculate the second term of the right hand side of (6), we use the Funk-Hecke theorem (see [10]):

$$\int_{S^{m-1}} f(< \underline{\omega}, \underline{\xi} >) S_k(\underline{\omega}) dS(\underline{\omega}) = A_{m-1} S_k(\underline{\xi}) \int_{-1}^{1} f(t)(1-t^2)^{(m-3)/2} P_k(t) dt,$$

with $S_k \in \mathcal{H}(k)$ a solid spherical harmonic of degree k and P_k the Legendre polynomial of degree k in $\mathbb{R}^m$.
Application of this theorem leads to

$$\begin{aligned}&\int_{S^{m-1}} \exp\left(-ir\rho < \underline{\omega}, \underline{\xi} >\right) \underline{\omega} dS(\underline{\omega}) \\ = \;& A_{m-1} \underline{\xi} \int_{-1}^{1} \exp\left(-ir\rho t\right)(1-t^2)^{(m-3)/2} P_1(t) dt.\end{aligned}$$

As

$$P_k(t) = \frac{k!(m-3)!}{(k+m-3)!} C_k^{(m-2)/2}(t)$$

and the Gegenbauer polynomials C_k^{λ} satisfy

$$C_k^{\lambda}(-x) = (-1)^k C_k^{\lambda}(x),$$

we obtain

$$\begin{aligned}&\int_{S^{m-1}} \exp\left(-ir\rho < \underline{\omega}, \underline{\xi} >\right) \underline{\omega} dS(\underline{\omega}) \\ = \;& -A_{m-1} \underline{\xi} \int_{-1}^{1} \exp\left(ir\rho t\right)(1-t^2)^{(m-3)/2} P_1(t) dt.\end{aligned}$$

This can be further calculated as

$$\begin{aligned}&\int_{S^{m-1}} \exp\left(-ir\rho < \underline{\omega}, \underline{\xi} >\right) \underline{\omega} dS(\underline{\omega}) \\ = \;& -iA_{m-1} 2^{m/2-1} \sqrt{\pi}\, \Gamma\left(\frac{m-1}{2}\right) \underline{\xi} \rho^{1-m/2} r^{1-m/2} J_{m/2}(\rho r).\end{aligned}$$

Consequently we obtain for the second term of the right hand side of (6):

$$\begin{aligned}&\frac{i}{2} \int_0^{+\infty} \exp(-r) r^{\alpha+2l+m-1} dr \int_{S^{m-1}} \exp\left(-ir\rho < \underline{\omega}, \underline{\xi} >\right) \underline{\omega} dS(\underline{\omega}) \\ = \;& 2^{m/2-2} \sqrt{\pi}\, \Gamma\left(\frac{m-1}{2}\right) A_{m-1} \underline{\xi} \rho^{1-m/2} \int_0^{+\infty} \exp(-r) r^{\alpha+2l+m/2} J_{m/2}(\rho r) dr \\ = \;& 2^{m/2-2} \sqrt{\pi}\, \Gamma\left(\frac{m-1}{2}\right) \Gamma(\alpha+2l+m+1) A_{m-1} \underline{\xi} \frac{\rho^{1-m/2}}{(1+\rho^2)^{(\alpha+2l+m/2+1)/2}} \\ & P_{\alpha+2l+m/2}^{-m/2}\left((1+\rho^2)^{-1/2}\right),\end{aligned}$$

where we have again used the assumption $\alpha > -m$.
Hence we finally obtain for the Fourier transform of the Clifford-Laguerre wavelets:

$$\begin{aligned}\mathcal{F}(\psi_{l,m,\alpha}(\underline{x}))(\underline{u}) \;=\;& (-i)^l 2^{m/2-1} \Gamma(\alpha+2l+m) \frac{|\underline{u}|^{1-m/2}}{(1+|\underline{u}|^2)^{(\alpha+2l+m/2+1)/2}} \underline{u}^l \\ & \left\{ \pi^{m/2} P_{\alpha+2l+m/2}^{1-m/2}\left((1+|\underline{u}|^2)^{-1/2}\right) + \frac{\sqrt{\pi}}{2} \Gamma\left(\frac{m-1}{2}\right) \right. \\ & \left. (\alpha+2l+m) A_{m-1} \frac{\underline{u}}{|\underline{u}|} P_{\alpha+2l+m/2}^{-m/2}\left((1+|\underline{u}|^2)^{-1/2}\right) \right\}. \qquad (7)\end{aligned}$$

5 The Clifford-Laguerre CWT

Take $f \in L_2(\mathbb{R}^m)$, then its Clifford-Laguerre CWT (CLCWT) is defined by

$$\begin{aligned} T_{l,m,\alpha} f(a,\underline{b},s) = F_{l,m,\alpha}(a,\underline{b},s) &= < \psi^{a,\underline{b},s}_{l,m,\alpha}, f > \\ &= \int_{\mathbb{R}^m} \overline{\psi}^{a,\underline{b},s}_{l,m,\alpha}(\underline{x}) f(\underline{x}) dV(\underline{x}) \end{aligned}$$

where, still for $\alpha > -m/2$ and $l > 0$, the continuous family of wavelets $\psi^{a,\underline{b},s}_{l,m,\alpha}(\underline{x})$ is given by

$$\psi^{a,\underline{b},s}_{l,m,\alpha}(\underline{x}) = \frac{1}{a^{m/2}} s \psi_{l,m,\alpha}\left(\frac{\overline{s}(\underline{x}-\underline{b})s}{a} \right) \overline{s},$$

with $a \in \mathbb{R}_+$, $\underline{b} \in \mathbb{R}^m$ and $s \in Spin(m)$, originating from the mother wavelet $\psi_{l,m,\alpha}$ by dilation, translation and spinor-rotation.

In agreement with the general representation theory of groups, we define the representation of $Spin(m)$ on the CLCWT as follows:

$$L(s') F_{l,m,\alpha}(a,\underline{b},s) = s' F_{l,m,\alpha}(a, \overline{s'}\underline{b}s', \overline{s'}s).$$

The above definition of the continuous family of wavelets leads to the commutation relation:

$$L(s') T_{l,m,\alpha} f(a,\underline{b},s) = T_{l,m,\alpha}(L(s')f)(a,\underline{b},s).$$

In other words, the use of the H-representation on the kernel of the wavelet transform is necessary in order to obtain the above $Spin(m)$-invariance of the wavelet transform.

Now it is clear that the CLCWT will map $L_2(\mathbb{R}^m)$ into a weighted L_2-space on $\mathbb{R}_+ \times \mathbb{R}^m \times Spin(m)$ for some weight function still to be determined. This weight function has to be chosen in such a way that the CLCWT is an isometry, or in other words that the Parseval formula should hold.

Introducing the inner product

$$\begin{aligned} & [F_{l,m,\alpha}, G_{l,m,\alpha}] \\ = \ & \frac{1}{C_{l,m,\alpha}} \int_{Spin(m)} \int_{\mathbb{R}^m} \int_0^{+\infty} \overline{F}_{l,m,\alpha}(a,\underline{b},s) G_{l,m,\alpha}(a,\underline{b},s) \frac{da}{a^{m+1}} dV(\underline{b}) ds, \end{aligned}$$

where ds stands for the Haar measure on $Spin(m)$, we search for the constant $C_{l,m,\alpha}$ in order to have the Parseval formula

$$< f, g >= [F_{l,m,\alpha}, G_{l,m,\alpha}]$$

fulfilled.

We find that this Parseval formula holds if we put (see [6])

$$C_{l,m,\alpha} = \int_{\mathbb{R}^m} \mathcal{F}(\psi_{l,m,\alpha}(\underline{x}))(\underline{u}) \overline{\mathcal{F}(\psi_{l,m,\alpha}(\underline{x}))}(\underline{u}) \frac{dV(\underline{u})}{|\underline{u}|^m},$$

where we have used the fact that $\mathcal{F}(\psi_{l,m,\alpha}(\underline{x}))\overline{\mathcal{F}(\psi_{l,m,\alpha}(\underline{x}))}$ is real valued.

The above relation defining the constant $C_{l,m,\alpha}$ is called the admissibility condition

for the Clifford-Laguerre wavelets, and the constant $C_{l,m,\alpha}$ involved is called the admissibility constant.
After substitution of the expression (7) for the Fourier transform, the admissibility constant takes the following form:

$$\begin{aligned}
& C_{l,m,\alpha} \\
= \; & 2^{m-2}\left(\Gamma(\alpha+2l+m)\right)^2\Bigg\{\pi^m \int_{\mathbb{R}^m} \frac{|\underline{u}|^{2l+2-m}}{(1+|\underline{u}|^2)^{(2\alpha+4l+m+2)/2}} \\
& \left(P^{1-m/2}_{\alpha+2l+m/2}((1+|\underline{u}|^2)^{-1/2})\right)^2 \frac{dV(\underline{u})}{|\underline{u}|^m} + \frac{\pi}{4}\left(\Gamma\left(\frac{m-1}{2}\right)(\alpha+2l+m)A_{m-1}\right)^2 \\
& \int_{\mathbb{R}^m} \frac{|\underline{u}|^{2l+2-m}}{(1+|\underline{u}|^2)^{(2\alpha+4l+m+2)/2}} \left(P^{-m/2}_{\alpha+2l+m/2}((1+|\underline{u}|^2)^{-1/2})\right)^2 \frac{dV(\underline{u})}{|\underline{u}|^m}\Bigg\}.
\end{aligned}$$

Introducing spherical co-ordinates

$$\underline{u} = \rho\underline{\xi}; \quad \rho = |\underline{u}|, \quad \underline{\xi} \in S^{m-1},$$

the first integral defining the admissibility constant can be simplified to

$$\begin{aligned}
& \int_{\mathbb{R}^m} \frac{|\underline{u}|^{2l+2-m}}{(1+|\underline{u}|^2)^{(2\alpha+4l+m+2)/2}} \left(P^{1-m/2}_{\alpha+2l+m/2}((1+|\underline{u}|^2)^{-1/2})\right)^2 \frac{dV(\underline{u})}{|\underline{u}|^m} \\
= \; & A_{m-1}\int_0^{+\infty} \frac{\rho^{2l-m+1}}{(1+\rho^2)^{(2\alpha+4l+m+2)/2}} \left(P^{1-m/2}_{\alpha+2l+m/2}((1+\rho^2)^{-1/2})\right)^2 d\rho.
\end{aligned}$$

As the associated Legendre functions of the first kind $P^{\mu}_{\nu}(x)$ with $\mu \neq 1, 2, 3, \ldots$ have the following behaviour near the singular point $+1$ (see for e.g. [11]):

$$\frac{2^{\mu/2}(1-x)^{-\mu/2}}{\Gamma(1-\mu)}$$

and as

$$P^{\mu}_{\nu}(0) = 2^{\mu}\pi^{-1/2}\cos\left(\frac{\pi}{2}(\nu+\mu)\right)\frac{\Gamma\left(\frac{1}{2}+\frac{\nu+\mu}{2}\right)}{\Gamma\left(1+\frac{\nu-\mu}{2}\right)},$$

the above integral is finite if we make the restriction $\alpha \neq -2l-2, -2l-4, -2l-6, \ldots$.
In an analogous manner, one obtains that the integral

$$\begin{aligned}
& \int_{\mathbb{R}^m} \frac{|\underline{u}|^{2l+2-m}}{(1+|\underline{u}|^2)^{(2\alpha+4l+m+2)/2}} \left(P^{-m/2}_{\alpha+2l+m/2}((1+|\underline{u}|^2)^{-1/2})\right)^2 \frac{dV(\underline{u})}{|\underline{u}|^m} \\
= \; & A_m\int_0^{+\infty} \frac{\rho^{2l-m+1}}{(1+\rho^2)^{(2\alpha+4l+m+2)/2}} \left(P^{-m/2}_{\alpha+2l+m/2}((1+\rho^2)^{-1/2})\right)^2 d\rho.
\end{aligned}$$

is finite, provided that $\alpha \neq -2l-1, -2l-3, -2l-5, \ldots$.
The above reasoning leads to the conclusion that the admissibility constant $C_{l,m,\alpha}$ is finite if we make the additional restriction $\alpha \neq -2l-1, -2l-2, -2l-3, -2l-4, \ldots$.

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