# Cohomological Hasse principle for the ring $\mathbb{F}_p((t))[[X,Y]]$

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**Abstract**

In this paper,we will prove the prime-to-p-part of a cohomological Hasse principle for the ring $A = \mathbb{F}_p((t))[[X,Y]]$. The proof is based on recent results by Fujiwara and Panin.

## 1 Introduction

By a global field, we mean a number field or a function field in one variable over a finite field. We start with the following classical exact sequence for a global field $K$:

$$0 \longrightarrow H^2(K, \mathbb{Z}/n(1)) \longrightarrow \bigoplus_v H^2(K_v, \mathbb{Z}/n(1)) \longrightarrow \mathbb{Z}/n \longrightarrow 0 \qquad (n \geq 1,) \tag{1.1}$$

where $v$ runs over all places of $K$, $K_v$ denotes the completion of $K$ at $v$ and $\mathbb{Z}/n(1)$ denotes the sheaf $\mu_n$ of $n$th root of unity. For any integer $i$, we denote $\mathbb{Z}/n(i) = \mu_n^{\otimes i}$.

A two dimensional global field is a field of transcendence degree one over $\mathbb{Q}$ or of transcendence degree two over $\mathbb{F}_p$. The exact sequence, corresponding to (1.1), for such a field has been established in [6] by Kato, who found that the group $H^2(K, \mathbb{Z}/n(1))$ must be replaced by $H^3(K, \mathbb{Z}/n(2))$. Besides, Kato pointed out the importance of the groups $H^{i+1}(K, \mathbb{Z}/n(i))$ $(i \geq 0)$ for a field of Kronecker dimension $i$.

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Let $X$ be a regular connected scheme of dimension $d$ over a finite field with function field $K$. Let $X_i$ denote the set of points of $X$ of dimension $i$. Kato ([6], Section 0) introduced the complex

$$C_n^0 \ldots \longrightarrow \underset{v\in X_j}{\oplus} H^{j+1}\left(k\left(v\right),\mathbb{Z}/n\left(j\right)\right) \to \ldots \to \underset{v\in X_1}{\oplus} H^2\left(k\left(v\right),\mathbb{Z}/n\left(1\right)\right) \longrightarrow \underset{v\in X_0}{\oplus} H^1\left(k\left(v\right),\mathbb{Z}/n\right)$$

and conjectured its exactness. Also, he proved the exactness of this complex whenever $X$ is a surface and obtained the following exact sequence:

$$0 \longrightarrow H^3\left(K,\mathbb{Z}/n\left(2\right)\right) \to \underset{v\in X_1}{\oplus} H^2\left(k\left(v\right),\mathbb{Z}/n\left(1\right)\right) \longrightarrow \underset{v\in X_0}{\oplus} H^1\left(k\left(v\right),\mathbb{Z}/n\right) \longrightarrow \mathbb{Z}/n \longrightarrow 0, \quad (1.2)$$

where $k(v)$ denotes the residue field of $X$ at $v$. Furthermore, Colliot-Thélène [1] and Saito [13] gave affirmative answers to this conjecture for three-dimensional global fields. The local version of the complex above was suggested by Kato (see [12], Section 5 by Saito). Actually, Saito studied the complex $C_n^0$ for 2-dimensional complete local rings having finite residue field. He showed that, if $A$ in particular is regular, then the following sequence is exact

$$0 \longrightarrow H^3\left(K,\mathbb{Z}/n\left(2\right)\right) \to \underset{v\in P}{\oplus} H^2\left(k\left(v\right),\mathbb{Z}/n\left(1\right)\right) \longrightarrow \mathbb{Z}/n \longrightarrow 0, \qquad (1.3)$$

where $P$ is the set of height one prime ideals of $A$ and $K$ is its fraction field. The case of 3-dimensional complete regular local rings of positive characteristic having finite residue field has been investigated by Matsumi in [7]. Indeed, the author proved the exactness of the following complex:

$$0 \longrightarrow H^4\left(K,\mathbb{Z}/n\left(3\right)\right) \to \underset{v\in (SpecA)_2}{\oplus} H^3\left(k\left(v\right),\mathbb{Z}/n\left(2\right)\right) \to \underset{v\in (SpecA)_1}{\oplus} H^2\left(k\left(v\right),\mathbb{Z}/n\left(1\right)\right) \longrightarrow \mathbb{Z}/n \longrightarrow 0, \quad (1.4)$$

for $n$ prime to char$(K)$. If at this point $A$ is not regular, then the kernel of the map

$$H^4\left(K,\mathbb{Z}/n\left(3\right)\right) \xrightarrow{\Psi_K} \underset{v\in (SpecA)_2}{\oplus} H^3\left(k\left(v\right),\mathbb{Z}/n\left(2\right)\right)$$

contains a subgroup of type $(\mathbb{Z}/n)^{r_1'(A)}$, where $r_1'(A)$ is calculated as the $\mathbb{Z}$-rank of the exceptional fiber graph of a resolution of $SpecA$ [3].

The present work deals with the study of the arithmetic Bloch-Ogus complex $C_n^0$ for 2-dimensional complete normal local rings of positive characteristic with a local field as residue field. Let $A$ be such a ring with fraction field $K$. Our two main results in this direction are the following.

**Theorem (Corollary 4)** *For all $\ell$ prime to the characteristic of $A$, the following sequence is exact*

$$0 \longrightarrow \pi_1^{c.s}(X)/\ell \longrightarrow H^4(K, \mathbb{Z}/\ell(3)) \rightarrow \underset{v\in P}{\oplus} H^3(k(v), \mathbb{Z}/\ell(2)) \longrightarrow \mathbb{Z}/\ell \longrightarrow 0.$$

*Here, the group* $\pi_1^{c.s}(X)$ *is the quotient group of* $\pi_1^{ab}(X)$ *that classifies completely split coverings of* $X$ (Definition 3 *below).*

If $A$ is regular, then, using [10] by Panin, we derive

**Theorem (Proposition 5)** *Let* $A = F_p((t))[[X,Y]]$. *Then for all* $\ell$ *prime to* $p$, *the following Hasse principle complex of* $A$ *is exact*

$$0 \longrightarrow H^4(K, \mathbb{Z}/\ell(3)) \longrightarrow \underset{v\in P}{\oplus} H^3(k(v), \mathbb{Z}/\ell(2)) \longrightarrow \mathbb{Z}/\ell \longrightarrow 0.$$

The proofs of both theorems above are based on "duality theory" as well as on the purity theorem of Fujiwara-Gabber. By "duality theory" we mean the following theorem, which we shall prove below.

**Theorem (Theorem 1)** *Let* $X = SpecA\backslash\{m\}$, where $m$ *is the unique maximal ideal of* $A$. *Then for all* $\ell$ *prime to the characteristic of* $A$ *the isomorphism*

$$H^5(X, \mathbb{Z}/\ell(3)) \simeq \mathbb{Z}/\ell$$

*and the perfect duality*

$$H^1(X, \mathbb{Z}/\ell) \times H^4(X, \mathbb{Z}/\ell(3)) \longrightarrow H^5(X, \mathbb{Z}/\ell(3)) \simeq \mathbb{Z}/\ell$$

*hold.*

Recently (see [4], observations after Corollary 7.1.7), Fujiwara established the formal base change theorem (II), and then observed that the absolute cohomological purity in equicharacteristic follows without the extra condition of the resolution of singularity. Let us state this observation as a theorem.

**Theorem of Fujiwara-Gabber** *Let* $T$ *be an equicharacteristic Noetherian excellent regular scheme, and* $Z$ *be a regular closed subscheme of codimension* $c$. *Then for an arbitrary natural number* $\ell$ *prime to char(*$T$*), the canonical isomorphism*

$$H_Z^i(T, \mathbb{Z}/\ell(j)) \simeq H^{i-2c}(Z, \mathbb{Z}/\ell(j-c))$$

*holds.*

## 2 Notations and Preliminaries

For an abelian group $M$ and a positive integer $n$, $M/n$ is the cokernel of the map $M \xrightarrow{n} M$. For a scheme $Z$, and a sheaf $\mathcal{F}$ over the étale site of $Z$, $H^i(Z, \mathcal{F})$ denotes the $i$th étale cohomology group. For a positive integer $\ell$ invertible on $Z$, $\mathbb{Z}/\ell(1)$ stands for the sheaf of $\ell$th roots of unity, and for an integer $i$, we denote $\mathbb{Z}/\ell(i) = (\mathbb{Z}/\ell(1))^{\otimes i}$.

A local field $k$ is said to be $d$-dimensional local if there exists a sequence of fields $k_i$ $(1 \leq i \leq d)$ such that
(i) each $k_i$ is a complete discrete valuation field having $k_{i-1}$ as the residue field of the valuation ring $O_{k_i}$ of $k_i$, and
(ii) $k_0$ is a finite field.

For such a field, and $\ell$ prime to Char$(k)$, we get (see [5], II §3.2 Proposition 1)

$$H^{d+1}(k, \mathbb{Z}/\ell(d)) \simeq \mathbb{Z}/\ell \tag{2.1}$$

and a perfect duality given by:

$$H^i(k, \mathbb{Z}/\ell(j)) \times H^{d+1-i}(k, \mathbb{Z}/\ell(d-j) \longrightarrow H^{d+1}(k, \mathbb{Z}/\ell(d)) \simeq \mathbb{Z}/\ell. \tag{2.2}$$

For a field $L$, let us denote by $K_i(L)$ the $i$th Quillen group [11]. It coincides with the $i$th Milnor group for $i \leq 2$. Recall that for such $i$ the cohomological symbol defined by Tate induces the isomorphism

$$h^i_{\ell,L} \quad K_iL/\ell \longrightarrow H^i(L, \mathbb{Z}/\ell(i)).$$

for all $\ell$ prime to char$(L)$ (note that for $i = 2$ we get the so-called Merkur'jev-Suslin Theorem).

Throughout, $A$ stands for a 2-dimensional complete normal local ring of positive characteristic with the 1-dimensional local field $k$ as residue field. The fraction field and the maximal ideal of $A$ are denoted by $K$ and $m$, respectively. If $A$ is regular, then $A$ is finite over $\mathbb{F}_p((t))[[X,Y]]$, where we use Cohen's structure theorem ([9],section 31). Finally, we put $P$ the set of height one prime ideals of $A$, and for $v \in P$ we set
$A_v$ : the henselization of $A$ at $v$,
$K_v$ : the fraction field of $A_v$,
$k(v)$ : its residue field.

## 3 The Kato'complex

In this section, we give the exact sequence for $A$ corresponding to (1.1), $A$ as defined in the previous section. To this end, we need a duality theorem for $X = SpecA \backslash \{m\}$, which will be deduced from Grothendieck's local duality. First, let us discuss the Grothendieck local duality. Let $B$ denote a $d$-dimensional normal complete local ring of positive characteristic, with maximal ideal $x'$. Now, assume firstly that the residue field of $B$ is separably closed (in other words $B$ is strictly local). Then, for $X' = SpecB \backslash \{x'\}$ and for any $\ell$ prime to char$(B)$, the Poincaré duality theory ([15],Exposé I,Remarque 4.7.17) says that there exists a trace isomorphism

$$H^{2d-1}(X', \mathbb{Z}/\ell(d)) \xrightarrow{\sim} \mathbb{Z}/\ell \tag{3.1}$$

and a perfect pairing

$$H^i(X', \mathbb{Z}/\ell) \times H^{2d-1-i}(X', \mathbb{Z}/\ell(d) \longrightarrow H^{2d-1}(X', \mathbb{Z}/\ell(d)) \simeq \mathbb{Z}/\ell. \tag{3.2}$$

Furthermore, we have (see [14], Exposé X, the last paragraph in the Introduction)

$$H^i\left(X', \mathbb{Z}/\ell(j)\right) = 0 \quad , \text{for all } \ i \geq 2d \tag{3.3}$$

Assume now that the residue field of $B$ is any field $k$. Let $k_s$ be a separable closure of $k$. The strict henselization $B^{sh}$ of $B$ (with respect to the separably closed extension $k_s$ of $k$) at the unique maximal ideal $x$ of $B$ is a strictly local ring. It coincides with the integral closure of $B$ in the maximal unramified extension $L^{ur}$ of the fraction field $L$ of $B$. If $x'$ is the maximal ideal of $B^{sh}$, we put $X' = SpecB^{sh} \backslash \{x'\}$ and $X = SpecB \backslash \{x\}$. Then the Galois group of $X'$ over $X$ is $\mathrm{Gal}(K^{ur}/K)$ which is isomorphic to $\mathrm{Gal}(k_s/k)$. Consequently, for any integer $j \geq 0$, we get the Hochschield-Serre spectral sequence (see [8], Remark 2.21)

$$E_2^{p,q} = H^p(k, H^q(X', \mathbb{Z}/\ell\,(j)) \Longrightarrow H^{p+q}(X, \mathbb{Z}/\ell\,(j)\,. \tag{3.4}$$

We are in position now to prove our duality theorem for $X = SpecA \backslash \{m\}$.

**Theorem 1.** *For all $\ell$ prime to the characteristic of $A$, the isomorphism*

$$H^5\left(X, \mathbb{Z}/\ell\,(3)\right) \simeq \mathbb{Z}/\ell \tag{3.5}$$

*and the perfect duality*

$$H^1\left(X, \mathbb{Z}/\ell\right) \times H^4\left(X, \mathbb{Z}/\ell(3)\right) \longrightarrow H^5\left(X, \mathbb{Z}/\ell\,(3)\right) \simeq \mathbb{Z}/\ell \tag{3.6}$$

*hold.*

*Proof.* Since $H^i\left(X', \mathbb{Z}/\ell\,(3)\right) = 0$ for all $i \geq 4$ (3.3) and $k$ is of cohomological dimension 2, the previous spectral sequence induces the isomorphism

$$H^5\left(X, \mathbb{Z}/\ell\,(3)\right) \simeq H^2(k, H^3(X', \mathbb{Z}/\ell\,(3)) \simeq H^2(k, \mathbb{Z}/\ell\,(1)) \simeq \mathbb{Z}/\ell,$$

which gives the first part of the theorem.

Next, we prove(3.6). The filtration of the group $H^4\left(X, \mathbb{Z}/\ell(3)\right)$ implies the exact sequence

$$0 \to E_\infty^{2,2} \longrightarrow H^4\left(X, \mathbb{Z}/\ell\,(3)\right) \longrightarrow E_\infty^{1,3} \longrightarrow 0$$

Since $E_2^{p,q} = 0$ for all $p \geq 3$, we find that $E_2^{1,3} = E_3^{1,3} = \ldots = E_\infty^{1,3}$. The same argument yields $E_3^{2,2} = E_4^{2,2} = \ldots = E_\infty^{2,2}$ and $E_3^{2,2} = Co\ker d_2^{0,3}$. Whence the following exact sequence:

$$0 \to Co\ker d_2^{0,3} \longrightarrow H^4\left(X, \mathbb{Z}/\ell\,(3)\right) \longrightarrow H^1(k, H^3(X', \mathbb{Z}/\ell\,(3)) \longrightarrow 0,$$

where $d_2^{0,3}$ is the map $H^0(k, H^3(X', \mathbb{Z}/\ell\,(3)) \to H^2(k, H^2(X', \mathbb{Z}/\ell\,(3))$.

Combining Tate's duality for $k$ and duality (3.2), we derive that the group $H^0(k, H^1(X', \mathbb{Z}/\ell))$ is dual to the group $H^2(k, H^2(X', \mathbb{Z}/\ell\,(3))$ and the group $H^2(k, H^0(X', \mathbb{Z}/\ell))$ is dual to the group $H^0(k, H^3(X', \mathbb{Z}/\ell\,(3))$. On the other hand, by the same argument as used in ([2],diagram 46), we get the commutative diagram

$$\begin{array}{ccccccc}
H^0(k, H^3(X', \mathbb{Z}/\ell\,(3)) & \times & H^2(k, H^0(X', \mathbb{Z}/\ell)) & \longrightarrow & H^2\left(k, \mathbb{Z}/\ell\,(1)\right) & \xrightarrow{\backsim} & \mathbb{Z}/\ell \\
\downarrow & & \uparrow & & \| & & \| \\
H^2(k, H^2(X', \mathbb{Z}/\ell\,(3)) & \times & H^0(k, H^1(X', \mathbb{Z}/\ell)) & \longrightarrow & H^2\left(k, \mathbb{Z}/\ell\,(1)\right) & \xrightarrow{\backsim} & \mathbb{Z}/\ell
\end{array}$$

given by the cup products and the spectral sequence (3.4). We infer that $Co\ker d_2^{0,3}$ is the dual of $Ker' d_2^{0,1}$, where $'d_2^{0,1}$ is the boundary map for the spectral sequence ((3.4), $j=0$)

$$'E_2^{p,q} = H^p(k, H^q(X', \mathbb{Z}/\ell) \Longrightarrow H^{p+q}(X, \mathbb{Z}/\ell).$$

Similarly, the group $H^1(k, H^3(X', \mathbb{Z}/\ell\,(3))$ is dual to $H^1(k, H^0(X', \mathbb{Z}/\ell))$. The desired duality is thus deduced from the following commutative diagram:

$$\begin{array}{ccccc}
0 \to Co\ker d_2^{0,3} & \longrightarrow & H^4\,(X, \mathbb{Z}/\ell\,(3)) & \longrightarrow & H^1(k, H^3(X', \mathbb{Z}/\ell\,(3)) \longrightarrow 0 \\
\downarrow \wr & & \downarrow \wr & & \downarrow \wr \\
0 \to \left(Ker' d_2^{0,1}\right)^{\vee} & \longrightarrow & \left(H^1\,(X, \mathbb{Z}/\ell)\right)^{\vee} & \longrightarrow & \left(H^1(k, H^0(X', \mathbb{Z}/\ell))\right)^{\vee} \longrightarrow 0
\end{array}$$

where $(M)^{\vee}$ denotes the dual $Hom(M, \mathbb{Z}/\ell)$ for any $\mathbb{Z}/\ell-$module $M$ and the bottom exact sequence is the dual of the well-known exact sequence

$$0 \longrightarrow' E_2^{1,0} \longrightarrow H^1\,(X, \mathbb{Z}/\ell) \longrightarrow Ker' d_2^{0,1} \longrightarrow 0.$$

The latter follows from the above spectral sequence. ∎

**Remark 2.** *If $v \in P$ then the duality (2.2)*

$$H^1\,(k(v), \mathbb{Z}/\ell) \times H^2\,(k(v), \mathbb{Z}/\ell(2)) \longrightarrow H^3\,(k, \mathbb{Z}/\ell\,(2)) \simeq\ \mathbb{Z}/\ell$$

*is compatible with the duality (3.6). In other words, the following diagram is commutative:*

$$\begin{array}{ccccccc}
H^1\,(X, \mathbb{Z}/\ell) & \times & H^4\,(X, \mathbb{Z}/\ell(3) & \longrightarrow & H^5\,(X, \mathbb{Z}/\ell\,(3)) & \xrightarrow{\backsim} & \mathbb{Z}/\ell \\
\downarrow i^* & & \uparrow i_* & & \uparrow i_* & & \| \\
H^1\,(k(v), \mathbb{Z}/\ell) & \times & H^2\,(k(v), \mathbb{Z}/\ell(2)) & \longrightarrow & H^3\,(k(v), \mathbb{Z}/\ell\,(2)) & \xrightarrow{\backsim} & \mathbb{Z}/\ell
\end{array}$$

*where $i^*$ is the map on $H^i$ induced from the map $y \longrightarrow X'$ and $i_*$ is the Gysin map. Commutativity of this diagram is obtained via the same argument (projection formula ([8], VI 6.5) and compatibility of traces ([8], VI 11.1)) as previously used in [2] in order to establish the commutative diagram in the proof of assertion ii) at page 791. This leads to the commutative diagram:*

$$\begin{array}{ccc}
H^2\,(k(v), \mathbb{Z}/\ell(2)) & \xrightarrow{i_*} & H^4\,(X, \mathbb{Z}/\ell(3) \\
\downarrow & & \downarrow \\
\left(H^1\,(k(v), \mathbb{Z}/\ell)\right)^{\vee} & \xrightarrow{(i^*)^{\wedge}} & \left(H^1\,(X, \mathbb{Z}/\ell)\right)^{\vee}
\end{array} \tag{3.7}$$

This theorem will be used to calculate the homologies of the arithmetic Bloch-Ogus complex associated to $A$. This complex is closely related to the quotient group of $\pi_1^{ab}\,(X)$ which classifies abelian c.s coverings of $X$.

**Definition 3.** *Let $Z$ be a noetherian scheme. A finite étale covering $f: W \to Z$ is called a c.s covering if for any closed point $z$ of $Z$, $z \times_Z W$ is isomorphic to a finite scheme-theoretic sum of copies of $z$. We denote $\pi_1^{c.s}\,(Z)$ the quotient group of $\pi_1^{ab}\,(Z)$ which classifies abelian c.s coverings of $Z$.*

For $X$ as in the above, the group $\pi_1^{c.s}(X)/\ell$ is the dual of the kernel of the map

$$H^1(X,\mathbb{Z}/\ell) \longrightarrow \prod_{v\in P} H^1(k(v),\mathbb{Z}/\ell) \tag{3.8}$$

(see [12], Section 2, Definition and sentence just below). Now, we are able to calculate the homologies of the Bloch-Ogus complex associated to the ring $A$.

**Corollary 4.** *For all $\ell$ prime to the characteristic of $A$, the following sequence is exact*

$$0 \longrightarrow \pi_1^{c.s}(X)/\ell \longrightarrow H^4(K,\mathbb{Z}/\ell(3)) \longrightarrow \underset{v\in P}{\oplus} H^3(k(v),\mathbb{Z}/\ell(2)) \longrightarrow \mathbb{Z}/\ell \longrightarrow 0$$

*Proof.* Consider the localization sequence on $X = SpecA\backslash\{m\}$

$$\ldots \to H^i(X,\mathbb{Z}/\ell(3)) \longrightarrow H^i(K,\mathbb{Z}/\ell(3)) \longrightarrow \underset{v\in P}{\oplus} H_v^{i+1}(X,\mathbb{Z}/\ell(3)) \to \ldots$$

Firstly, for any $v\in P$, we have the isomorphisms

$$H_v^i(X,\mathbb{Z}/\ell(3)) \simeq H_v^i(SpecA_v,\mathbb{Z}/\ell(3)) \qquad \text{by excision.}$$

Secondly, we can apply the purity theorem of Fujiwara-Gabber (see Introduction) for $Z = v, T = SpecA_v$ to obtain the isomorphisms

$$H_v^4(SpecA_v,\mathbb{Z}/\ell(3)) \simeq H^2(k(v),\mathbb{Z}/\ell(2)),$$
$$H_v^5(SpecA_v,\mathbb{Z}/\ell(3)) \simeq H^3(k(v),\mathbb{Z}/\ell(2))$$

But then

$$H_v^4(X,\mathbb{Z}/\ell(3)) \simeq H^2(k(v),\mathbb{Z}/\ell(2))$$

and

$$\begin{array}{ll} H_v^5(X,\mathbb{Z}/\ell(3)) & \simeq H^3(k(v),\mathbb{Z}/\ell(2)) \\ & \simeq \mathbb{Z}/\ell, \quad k(v) \text{ is a two-dimensional local field} \\ H^5(K,\mathbb{Z}/\ell(3)) = 0, & cd_\ell(K) < 4 \end{array}$$

Hence we get the exact sequence

$$\begin{array}{l} \underset{v\in P}{\oplus} H^2(k(v),\mathbb{Z}/\ell(2)) \xrightarrow{g} H^4(X,\mathbb{Z}/\ell(3)) \longrightarrow H^4(K,\mathbb{Z}/\ell(3)) \\ \qquad \longrightarrow \underset{v\in P}{\oplus} H^3(k(v),\mathbb{Z}/\ell(2)) \longrightarrow H^5(X,\mathbb{Z}/\ell(3)) \longrightarrow 0 \end{array}$$

By the previous theorem, we get the isomorphism $H^5(X,\mathbb{Z}/\ell(3)) \simeq \mathbb{Z}/\ell$, which yields the exact sequence

$$0 \longrightarrow Co\ker g \longrightarrow H^4(K,\mathbb{Z}/\ell(3)) \longrightarrow \underset{v\in P}{\oplus} H^3(k(v),\mathbb{Z}/\ell(2)) \longrightarrow \mathbb{Z}/\ell \longrightarrow 0$$

On the other hand, the commutative diagram (3.7) implies that $Co\ker g$ is dual to the kernel of the map

$$H^1(X,\mathbb{Z}/\ell) \longrightarrow \prod_{v\in P} H^1(k(v),\mathbb{Z}/\ell),$$

which is equal to $\pi_1^{c.s}(X)/\ell$ (3.8) ∎

Next, assume that the ring $A$ is regular. We shall prove that the group $\pi_1^{c.s}(X)/\ell$ vanishes.

**Proposition 5.** *Let $A = \mathbb{F}_p((t))[[X,Y]]$. Then for all $\ell$ prime to $p$, the following Hasse principle complex of $A$ is exact:*

$$0 \longrightarrow H^4(K, \mathbb{Z}/\ell(3)) \longrightarrow \underset{v\in P}{\oplus} H^3(k(v), \mathbb{Z}/\ell(2)) \longrightarrow \mathbb{Z}/\ell \longrightarrow 0.$$

*Proof.* We only have to prove the injectivity of the map

$$\Psi_K: \quad H^4(K, \mathbb{Z}/\ell(3)) \longrightarrow \underset{v\in P}{\oplus} H^3(k(v), \mathbb{Z}/\ell(2)).$$

Let $q$ be an integer and consider the sheaf $\mathcal{H}^q(\mathbb{Z}/\ell(3))$ on $SpecA$, the Zariskien sheaf associated to the presheaf $U \longrightarrow H^q(U, \mathbb{Z}/\ell(3))$. The recent works of Panin [10] allows us to see that the cohomology of this sheaf is calculated as the homology of the Bloch-Ogus complex

$$H^q(K, \mathbb{Z}/\ell(3)) \longrightarrow \underset{v\in P}{\oplus} H^{q-1}(k(v), \mathbb{Z}/\ell(2)) \longrightarrow H^{q-1}(k(m), \mathbb{Z}/\ell(1)).$$

Therefore the group $Ker\Psi_K$ is identified with the group $H^0((SpecA)_{Zar}, \mathcal{H}^4(\mathbb{Z}/\ell(3)))$.

On the other hand, the Bloch-Ogus spectral sequence

$$H^p((SpecA)_{Zar}, \mathcal{H}^q(\mathbb{Z}/\ell(3))) \Rightarrow H^{p+q}(SpecA, \mathbb{Z}/\ell(3))$$

gives the exact sequence

$$\begin{aligned} H^4(SpecA, \mathbb{Z}/\ell(3)) &\longrightarrow H^0\left((SpecA)_{Zar}, \mathcal{H}^4(\mathbb{Z}/\ell(3))\right) \\ &\longrightarrow H^2\left((SpecA)_{Zar}, \mathcal{H}^3(\mathbb{Z}/\ell(3))\right) \to H^5(SpecA, \mathbb{Z}/\ell(3)) \end{aligned}$$

As the ring $A$ is henselian, we obtain the isomorphism

$$H^i(SpecA, \mathbb{Z}/\ell(3)) \simeq H^i(Speck, \mathbb{Z}/\ell(3)), i \geq 0.$$

But the groups $H^4(Speck, \mathbb{Z}/\ell(3))$ and $H^5(Speck, \mathbb{Z}/\ell(3))$ vanish, because the cohomological dimension of $k$ is 2. Hence, we have the isomorphism

$$H^0\left((SpecA)_{Zar}, \mathcal{H}^4(\mathbb{Z}/\ell(3))\right) \xrightarrow{\sim} H^2\left((SpecA)_{Zar}, \mathcal{H}^3(\mathbb{Z}/\ell(3))\right),$$

which means that $Ker\Psi_K$ is isomorphic to the Cokernel of the map

$$\underset{v\in P}{\oplus} H^2(k(v), \mathbb{Z}/\ell(2)) \longrightarrow H^1(k(m), \mathbb{Z}/\ell(1))$$

Moreover, by the theorem of Merkur'jev-Suslin and by Kummer's theory, the latter map can be written as

$$\underset{v\in P}{\oplus} K_2(k(v))/\ell \longrightarrow k(m)^\times/\ell,$$

where we use the commutative diagram

$$\begin{array}{ccc} \bigoplus_{v\in P} K_2\left(k\left(v\right)\right)/\ell & \longrightarrow & k\left(m\right)^{\times}/\ell \\ \downarrow\wr & & \downarrow\wr \\ \bigoplus_{v\in P} H^2\left(k\left(v\right),\mathbb{Z}/\ell\left(2\right)\right) & \longrightarrow & H^1\left(k\left(m\right),\mathbb{Z}/\ell\left(1\right)\right) \end{array}$$

The surjectivity of the map

$$\bigoplus_{v\in P} H^2\left(k\left(v\right),\mathbb{Z}/\ell\left(2\right)\right) \longrightarrow H^1\left(k\left(m\right),\mathbb{Z}/\ell\left(1\right)\right)$$

is then deduced from the exactness of the Gersten-Quillen complex ([10],theorem A):

$$K_3\left(A\right)/\ell \longrightarrow K_3\left(K\right)/\ell \longrightarrow \bigoplus_{v\in P} K_2\left(k\left(v\right)\right)/\ell \longrightarrow k\left(m\right)^{\times}/\ell \longrightarrow 0$$

■

**Remark 6.** *Following Kato and Saito [12], we can construct the reciprocity map*

$$\phi_K \qquad H^1\left(K,\mathbb{Z}/\ell\right) \longrightarrow Hom\left(C_K,\mathbb{Z}/\ell\right)$$

*for the fraction field* $K$ *of the ring* $A$*, where* $C_K$ *is an ideal class group associated to* $K$*. The fact that* $\pi_1^{c.s}\left(X\right)$ *vanishes for a regular ring implies the injectivity of the reciprocity map.*

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