BOOK REVIEWS

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 24, Number 2, April 1991
©1991 American Mathematical Society
0273-0979/91 $1.00 + $.25 per page

Algebraic structures of bounded symmetric domains, by I. Satake. Iwanami Shoten and Princeton University Press, 1980, x + 317 pp. ISBN 0-691-08271-5

This is an important book because it is the first one discussing the general theory of bounded symmetric domains under all its important aspects. This may sound surprising given the age and importance of the subject; the explanation lies probably in its difficulty and in its rich and complicated relations with other fields.

To describe the subject briefly, consider a bounded domain D in $\mathbf{C}^n$. Let $\mathrm{Aut}(D)$ denote the group of all holomorphic bijections of D onto itself; this is a Lie group (e.g., because it is a closed subgroup of the isometry group of D in the Bergman metric). D is said to be *homogeneous* if for all z, w in D, there is a g in $\mathrm{Aut}(D)$ with $g(z) = w$. D is *symmetric* if there is always a g with $g(z) = w$ *and* $g(w) = z$. (With the Bergman metric such a D is a Riemannian, even Hermitian, symmetric space; hence $\mathrm{Aut}(D)$ is semisimple... .) When $n = 1$, the unit disc and every simply connected domain are symmetric. When $n > 1$, there is no Riemann mapping theorem, and homogeneous domains become quite rare. Symmetric domains are even rarer: For every n, there are only finitely many nonequivalent ones (due to a similar situation for semisimple Lie groups). They are completely classified. There are five (or four depending on how one counts) infinite classes called the classical domains; they correspond to certain Lie groups of classical type, and there are two "exceptional domains," one in $\mathbf{C}^{16}$ and one in $\mathbf{C}^{27}$. E. Cartan did the classification in 1935 up to the proof of existence of the exceptional domains; this was given by Harish-Chandra in 1956 by a general method, independent of classification.

Up to this point everything is in Helgason's books, already in the 1962 classic, and also in its new several-volume version. In Satake's book all this is only briefly summarized.

Just as the disc in $\mathbf{C}$ can be regarded as a domain on the Riemann sphere, every Cartan domain (= bounded symmetric domain) can be imbedded into a compact Hermitian symmetric space (A. Borel). One can define a Cayley transformation, describe a half-plane-like version of the domain, and at the same time describe its exact boundary structure. This was done for the classical domains by Pyatetski-Šapiro, and in general by J. A. Wolf and the reviewer. These matters are treated in detail in the present book. The author has his own way of doing this, based on the study of certain types of homomorphisms of Lie algebras.

To continue with history, in the 1960s M. Koecher and his school developed an alternative way of studying the Cartan domains, based on Jordan algebras and on the somewhat more general category of Jordan triple systems. This is an independent approach since it permits building up the theory described above (except perhaps the Borel imbedding) without any differential geometry and Lie theory. It also gives a number of explicit formulas (i.e., explicit in terms of quantities intrinsic to the Jordan triple system). A great merit of Satake's book is that it also describes this approach and establishes its precise relations to the standard one.

What does one do with the Cartan domains? The first answer is that one studies holomorphic functions and generalized harmonic functions on them, integral representations, boundary behavior, Hilbert spaces of holomorphic functions which carry unitary representations (Harish-Chandra's "holomorphic discrete series") of certain semisimple Lie groups, and various other function spaces and operators. The unit ball in $\mathbf{C}^n$ is, after all, the simplest kind of Cartan domain; generalizing things known for the unit ball is sometimes uninteresting, sometimes impossible, but sometimes genuinely interesting. Such questions are not considered in the book, but the bibliography contains many references to them.

A second and even more important answer is that one looks at discrete subgroups Γ of $\mathrm{Aut}(D)$, which have a fundamental domain of finite Bergman volume in D. (By results of Selberg and Hano, one sees, by the way, that such a Γ can exist only if

D is symmetric; homogeneity is not enough.) This is the beginning of the theory of automorphic functions. In one variable, if Γ is the modular group, $\Gamma\backslash D$ gives, through the theory of the Weierstrass p-function, the natural parametrization of the possible complex structures on the 2-torus. In the general case there are similar families of Abelian varieties. These are Kuga's fiber varieties, which are constructed in the book in the author's own way, and their algebraicity is proved.

As to the organization of the book, it contains five chapters of roughly equal length. The first two summarize the essential facts about algebraic groups and semisimple Lie groups with a few proofs, and give the Jordan algebra prerequisites with concise proofs. The third is about Cayley transforms and boundary structure, and the fourth about equivariant holomorphic maps, culminating in the results about Kuga's fiber varieties. Chapter 5 gives more detail about the Lie algebra of $\mathrm{Aut}(D)$ for the half-plane version of D; it includes, up to a point, nonsymmetric domains. There is also an Appendix, where the classical domains are explicitly constructed.

The book is not easy to read because of its conciseness and because of the many prerequisites, such as linear algebraic groups, that it constantly uses. On the other hand, it is very carefully written and organized; everything is in its place. It has a good index and index of notations, and a very detailed bibliography. Also, despite the unfortunate circumstance that this review is written so long after the book's publication, it is still the only book on this subject.

ADAM KORANYI
LEHMAN COLLEGE (CUNY)

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 24, Number 2, April 1991
©1991 American Mathematical Society
0273-0979/91 $1.00 + $.25 per page

Numerical methods for initial value problems in ordinary differential equations, by S. O. Fatunla. Academic Press, London, 265 pp., $44.50. ISBN 0-12-249930-1

For many years the textbooks by Gear [2] and by Lambert [7] and the summer school proceedings edited by Hall and Watt [4]