BOOK REVIEWS

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An introduction to intersection homology theory, by Frances Kirwan. Pitman Research Notes in Mathematics Series 187, Longman Scientific and Technical, Harlow, Essex, CM20 2JE, United Kingdom, 1988, vi + 169 pp., $47.95. ISBN 0-582-02879-5 and ISBN 0-470-21198-9

1. Introduction

Intersection homology theory is a magnificent new tool for working with the common singular spaces, and it has engendered profound results in topology, analysis, arithmetic algebraic geometry, $\mathcal{D}$-module theory, and representation theory. When the spaces are smooth, the theory agrees with ordinary homology theory. However, when the spaces are singular, then it, unlike the ordinary theory, continues to satisfy Poincaré duality and the Künneth formula. Moreover, when the spaces are projective algebraic varieties in any characteristic, then it continues to exist, and it satisfies the two Lefschetz theorems as well. There is a corresponding L^2-deRham–Hodge theory, and when the spaces are complex projective varieties, then there is a pure Hodge structure. During the fifteen years that have elapsed since its discovery, intersection homology theory has stimulated the frenetic efforts of an unprecedented and ever increasing number of mathematicians, including many of today's most gifted; they have done some of the most important mathematics of the century.

The book under review does not emphasize the historical development of intersection homology theory. The following sections may therefore help to introduce the contents of the book and put them in perspective. These sections are summarized from the author's fuller account.[1]

2. Discovery

Intersection homology theory was discovered in the fall of 1974 by Mark Goresky and Robert MacPherson. They were looking for a theory of characteristic numbers for singular spaces X. Previously, MacPherson and others had found several characteristic homology classes, but homology classes cannot be multiplied. So, with the second homology operations in mind, Goresky and MacPherson hoped to find certain "intersectable" classes, whose intersection product would be well defined modulo certain "indeterminacy" classes. Moreover, they knew how to view cohomology groups on spaces with "Whitney" stratifications as the homology groups of certain "geometric" cycles, which can be made transverse to each other and to each stratum.

Goresky and MacPherson relaxed the transversality condition on the cycles by allowing them to deviate from dimensional transversality to each stratum of codimension k, for each $k \geq 2$ (by hypothesis there are no strata of codimension 1), within a tolerance specified by a function $\overline{p}(k)$, which they called the *perversity.* And they allowed the homologies to deviate in the same way. Thus, for each $\overline{p}(k)$ and each i, they obtained a new group $IH_i^{\overline{p}}(X)$, and eventually they called it the "intersection homology group."

The perversity $\overline{p}(k)$ is required to satisfy the condition $\overline{p}(2) = 0$; consequently, the i-cycle lies mostly in the nonsingular part of X, where it is orientable. If X is compact, then $IH_i^{\overline{p}}(X)$ is finitely generated, and there are intersection pairings,

$$IH_i^{\overline{p}}(X) \times IH_j^{\overline{q}}(X) \to IH_{i+j-n}^{\overline{p}+\overline{q}}(X).$$

The (topological) normalization map $X' \to X$ induces an isomorphism,

$$IH_i^{\overline{p}}(X') \xrightarrow{\sim} IH_i^{\overline{p}}(X).$$

If X is normal, then $IH_i^{\overline{p}}(X)$ ranges from the ordinary cohomology group where $\overline{p}(k) = 0$ for all k to the ordinary homology

[1] *The development of intersection homology theory,* in A Century of Mathematics in America. Part II, Amer. Math. Soc., 1989. Remark: That account is in part an interpretation of the retrospections of those interviewed, and not necessarily in accord with the retrospections of others. The author is grateful to Clint McCrory, Masaki Kashiwara, and Pierre Schapira for taking the trouble of reminding him of that fact.

groups where $\bar{p}(k) = k - 2$ for all k; if X is also compact, then the pairing generalizes the usual cup and cap products. Moreover, the theory extends to noncompact X when cycles with compact supports are used.

Goresky and MacPherson realized that just as cohomology groups and homology groups are dually paired, so too the intersection homology groups of complementary dimension $(i + j = n)$ and complementary perversity $(\bar{p}(k) + \bar{q}(k) = k - 2)$ are dually paired: Poincaré duality holds! Sullivan's 1970 problem was solved: *If X is compact, has dimension $4l$, and has only even codimensional strata, then the middle perversity group $IH_{2l}^{\bar{m}}(X)$, where $\bar{m}(k) := \lfloor \frac{k-2}{2} \rfloor$, carries a nondegenerate bilinear form, whose signature is invariant under cobordisms with even codimensional strata.* In the summer of 1975, Goresky and MacPherson discovered that the growth condition $\bar{p}(k) \le \bar{p}(k+1) \le \bar{p}(k) + 1$ implies that the groups $IH_i^{\bar{p}}(X)$ are invariant under restratification. During the academic year 1975–1976, they settled on working in the pl-category, and benefited greatly from Clint McCrory's expertise on pl-transversality.

3. L^2-COHOMOLOGY

During the winter of 1975–1976, Jeff Cheeger found, independently of Goresky and MacPherson, a cohomology theory satisfying Poincaré duality for essentially the same class of spaces X. Such a closed oriented pseudomanifold X carries natural piecewise flat metrics, and Cheeger formed the L^2-cohomology groups of the incomplete Riemannian manifold U obtained by discarding all the simplices of codimension 2 or more. Those are the cohomology groups $H_{(2)}^i(U)$ of the complex of real differential forms ω on U such that

$$\int_U \omega \wedge * \omega < \infty \quad \text{and} \quad \int_U d\omega \wedge * d\omega < \infty.$$

Cheeger found that Poincaré duality could be verified directly or derived formally, in essentially the same way as in the smooth case, from the action of the $*$-operator on the harmonic forms of the associated Hodge theory—in fact, the full Hodge theory holds—given a certain condition. The condition was later seen to hold whenever X has a stratification by strata of even codimension. The theory automatically works also if X is equipped with any metric that, on U, is quasi-isomorphic to the previous one; then X is said to have 'conical' or 'conelike' singularities. Cheeger eventually proved that he found a de Rham–Hodge theory dual to

Goresky and MacPherson's combinatorial theory for the middle perversity.

Cheeger's discovery was an extraordinary byproduct of his work on his proof of the Ray–Singer conjecture. Cheeger's analytic methods in intersection homology theory yielded the first proof of the Künneth formula (the combinatorial approach had failed, because the product of two middle-allowable cycles is seldom middle allowable). Cheeger's methods have also yielded the only known explicit local formulas for the L-class and a vanishing theorem for the intersection homology groups of a pseudomanifold of positive curvature. Moreover, the general methods themselves have also had significant applications to other theories, including index theory for families of Dirac operators, the theory surrounding Witten's global anomaly formula, and diffraction theory.

In the summer of 1977, Cheeger and MacPherson chatted. However they considered not the conical metric of a triangulation, but the Kähler metric of a complex projective variety X with nonsingular part U. They conjectured that (1) *the L^2-cohomology group $H^i_{(2)}(U)$ is always dual to the intersection homology group $IH_i(X)$ under the integration pairing and* (2) *the standard consequences of Hodge theory—including the Hodge decomposition, the hard Lefschetz theorem, and the Hodge index theorem—are valid.* Those conjectures were published in a 1980 paper of Cheeger's. With Goresky's help, Cheeger and MacPherson developed the conjectures further, supported them with examples, and published them in 1982 in a joint article, which has inspired many people.

4. A FORTUITOUS ENCOUNTER

At a 1976 Halloween party, MacPherson introduced Deligne to intersection homology theory, leading Deligne to write down his celebrated formula,

$$IH_i^{\bar{p}}(X) = H^{n-i}(\mathbf{IC}^{\bullet}_{\bar{p}}(X)) \quad \text{where } n := \dim(X).$$

The formula expresses $IH_i^{\bar{p}}(X)$ as the hypercohomology group of the following complex of sheaves:

$$\mathbf{IC}^{\bullet}_{\bar{p}}(X) := \tau_{\leq \bar{p}(n)} \mathbf{R}i_{n*} \cdots \tau_{\leq \bar{p}(2)} \mathbf{R}i_{2*} \mathbf{C}_{X-X_2}$$

where X_k is the union of all strata of codimension k or more, where $\mathbf{C}_{X-X_2}$ is the complex consisting of the constant sheaf of complex numbers concentrated in degree 0, where i_k is the inclusion of $X - X_k$ into $X - X_{k+1}$, and where $\tau_{\leq k}$ is the truncation functor that kills the stalk cohomology in degree above k.

The complex $\mathbf{IC}^{\bullet}_{\bar{p}}(X)$ is, however, well defined only in the *derived category*—the category constructed out of the category of complexes up to *homotopy equivalence*, by requiring a map of complexes to be an isomorphism (to possess an inverse) if and only if it induces an isomorphism on the cohomology sheaves.

In a seminar during the academic year 1977–1978, Goresky and MacPherson worked out the first proof of Deligne's formula, but it was complicated, and they decided to streamline it. They made steady progress during the next year, 1979–1980: They found several axiomatic characterizations of $\mathbf{IC}^{\bullet}_{\bar{p}}(X)$, and used them to prove Deligne's formula and the *topological* invariance of $IH_i^{\bar{p}}(X)$ (its independence of the stratification and the pl-structure), and to reprove the Künneth formula for the middle perversity $\bar{m}$. They also adapted Grothendieck's sheaf theoretic proof of the Lefschetz hyperplane theorem for $\bar{m}$. (The year before, while working on their new stratified Morse theory, they found they could adapt Thom's argument to give the first proof of the Lefschetz theorem.)

5. The Kazhdan–Lusztig conjecture

In 1978 David Kazhdan and George Lusztig found a new construction of Tony Springer's l-adic representation of the Weyl group W of a semisimple algebraic group over a finite field. The representation module has two natural bases, and they tried to identify the transition matrix. Thus they were led to define, by an effective combinatorial procedure, some new polynomials $P_{y,w}$ with integer coefficients indexed by the pairs of elements $y, w \in W$, and $y \leq w$, for any Coxeter group W.

The two bases reminded Kazhdan and Lusztig of the two natural bases of the Grothendieck group of the (Bernstein–Gelfand–Gelfand) category $\mathscr{O}_{\text{triv}}$ of certain infinite dimensional representations of a complex semisimple Lie algebra $\mathfrak{g}$: the basis formed by the Verma modules M_λ and that by the simple modules L_μ. Putting aside their work on the Springer representation, they focused on the transition matrix between the M_λ and L_μ, and were led to formulate the following conjecture: *In the Grothendieck group,*

$$L_{-\rho w-\rho} = \sum_{y\leq w} (-1)^{l(w)-l(y)} P_{y,w}(1) M_{-\rho y-\rho},$$

or equivalently,

$$M_{\rho w-\rho} = \sum_{w\leq y} P_{w,y}(1) L_{\rho y-\rho},$$

where, as usual, ρ *is half the sum of the positive roots, and* $l(w)$ *is the length of* w*, the dimension of the Schubert variety* X_w. That formulation was taken from one of Lusztig's papers; the original conjecture appeared in a joint paper, which was received for publication on March 11, 1979.

Kazhdan and Lusztig felt that "$P_{y,w}$ can be regarded as a measure for the failure of local Poincaré duality" on the Schubert variety X_w in a neighborhood of a point of the Bruhat cell B_y. At MacPherson's suggestion they wrote to Deligne. Deligne responded from Paris on April 20, 1979 with a famous seven-page letter. In it, he observed that the sheaf-theoretic approach works equally well for a projective variety X over the algebraic closure of a finite field with $q := p^e$ elements, with the étale topology and sheaves of $\mathbf{Q}_l$-vector spaces, $l \neq p$. The strata must be smooth and equidimensional, but it is unnecessary that the normal structure of X be locally trivial in any particular sense along each stratum; it suffices that the stratification be fine enough so that all the sheaves involved are locally constant on each stratum. Deligne stated that the Lefschetz fixed-point formula is valid for the Frobenius endomorphism $\phi_q : X \to X$, which raises the coordinates of a point to the q th power. The fixed-points x of ϕ_q are simply the points $x \in X$ with coordinates in $\mathbf{F}_q$, and the formula expresses their number (counted with appropriate multiplicities when they are singular points) as the alternating sum of the traces of ϕ_q on the $IH^i(X)$; here, as is conventional, the perversity is omitted when it is the middle perversity.

Deligne wrote that he could not prove the following form of "purity": *For every fixed-point* x *and for every* i*, the eigenvalues of* ϕ_q *on the stalk at* x *of the sheaf* $\mathbf{H}^i(\mathbf{IC}^\cdot(X))$ *are algebraic numbers whose complex conjugates all have absolute value at most* $q^{i/2}$. However, if purity holds, then so will the following two theorems, which Kazhdan and Lusztig had asked about: (1) (Weil–E. Artin-Riemann hypothesis) *The eigenvalues of* ϕ_q *on* $IH^i(X)$ *are algebraic numbers whose complex conjugates are all of absolute value* $q^{i/2}$; (2) (hard Lefschetz theorem) *If* $[H] \in H^2(\mathbf{P}^N)$ *denotes the fundamental class of a hyperplane* H *in the ambient projective space, then for all* i*, intersecting* i *times yields an isomorphism,*

$$(\cap[H])^i : IH^{d-i}(X) \xrightarrow{\sim} IH^{d+i}(X) \quad \text{where } d := \dim(X).$$

Kazhdan and Lusztig then proved purity directly in the case of the Schubert varieties X_w by exploiting the geometry. In fact, they proved the following stronger theorem: *The sheaf* $\mathbf{H}^{2j+1}\mathbf{IC}^\cdot(X_w)$

is zero; on the stalk at a fixed point, $\mathbf{H}^{2j}\mathbf{IC}^{\cdot}(X)_x$, *the eigenvalues of* ϕ_q *are algebraic numbers whose complex conjugates all have absolute value exactly* q^j. On the basis of those theorems, Kazhdan and Lusztig proved their main theorem: *The coefficients of* $P_{y,w}$ *are positive; in fact,*

$$\sum_j \dim(\mathbf{H}^{2j}\mathbf{IC}^{\cdot}(X_w)_y)q^j = P_{y,w}(q),$$

where the subscript y *indicates the stalk at the base point of* B_y.

6. $\mathscr{D}$-MODULES

By good fortune, the theory needed to establish the Kazhdan–Lusztig conjecture was actively being developed in the late 1970s. In fact, the theory was needed as much for its spirit as for its results. It is a sophisticated modern theory of linear partial differential equations on a smooth complex algebraic variety X. It is often called *$\mathscr{D}$-module theory,* because it involves sheaves of modules $\mathscr{M}$ over the sheaf of (noncommutative) rings of holomorphic linear partial differential operators of finite order, $\mathscr{D} := \mathscr{D}_X$.

A major theme in $\mathscr{D}$-module theory is the modern *Riemann–Hilbert problem,* the definitive generalization of David Hilbert's twenty-first problem. Hilbert put the original problem as follows: *Show that there always exists a linear differential equation of Fuchsian class with given singular points and monodromic group.* He added that it is "an important problem, one which very likely Riemann himself may have had in mind." In the fall of 1969, Deligne generalized the setting from the complex projective line to a smooth complex algebraic variety X of arbitrary complex dimension d; the importance of Deligne's contribution cannot be overestimated; it inspired and supported all the subsequent advances.

Between 1975 and 1980, the modern Riemann–Hilbert problem was gradually formulated and solved analytically, but somewhat differently by Masaki Kashiwara and by Zoghman Mebkhout; unfortunately, there is some controversy over priority. In the fall of 1980, Alexandre Beilinson and Joseph Bernstein developed a simpler and purely algebraic treatment, which is more than sufficient for the proof of the Kazhdan–Lusztig conjecture.

The problem is to prove this theorem: *The "Riemann–Hilbert correspondence"*

$$\mathscr{M} \mapsto \mathrm{de\,R}(\mathscr{M})$$

is an equivalence of derived categories, which commutes with direct image, inverse image, exterior tensor product, and duality. The source of the correspondence is the derived category of complexes $\mathscr{M}$ of $\mathscr{D}$-modules whose cohomology sheaves $\mathscr{H}^i(\mathscr{M})$ are regular holonomic $\mathscr{D}$-modules; those complexes generalize the differential equations of Fuchsian class. The essential image is the *constructible derived category,* the derived category of bounded complexes of sheaves of complex vector spaces with constructible cohomology sheaves; those complexes generalize the spaces of solutions with given monodromy action.

The Kazhdan–Lusztig conjecture was proved during the summer and fall of 1980 independently and in essentially the same way by Beilinson and Bernstein in Moscow and by Jean-Luc Brylinski and Kashiwara in Paris. There are two main lemmas, which concern the flag manifold X: (1) *The functor* $M \mapsto \mathscr{D}_X \otimes M$ *embeds the category* $\mathscr{O}_{\mathrm{triv}}$ *in the category of regular holonomic* $\mathscr{D}_X$*-modules.* (2) *If* $d := \dim_{\mathbf{C}}(X)$, *these formulas hold,*

$$\operatorname{de R}(\mathscr{D}_X \otimes M_{-\rho w-\rho}) = \mathbf{C}_w[l(w) - d]$$
$$\operatorname{de R}(\mathscr{D}_X \otimes L_{-\rho w-\rho}) = \mathbf{IC}^{\cdot}(X_2)[l(w) - d].$$

The conjecture follows directly. Indeed, consider the *index,*

$$\chi_w(M) := \sum_i (-1)^i \dim_{\mathbf{C}} \mathbf{H}^i(\operatorname{de R}(\mathscr{D}_X \otimes M))_w.$$

If δ_{wy} is the Kronecker function, the first formula and additivity yield

$$\chi_w(M_y) = (-1)^{l(w)-d}\delta_{wy} \quad \text{and} \quad M = \sum_y (-1)^{d-l(y)} \chi_y(M) M_y.$$

Finally, the second formula of (2) yields the conjecture.

7. Perverse sheaves

In mid-September 1980, Beilinson, Bernstein, and Deligne got together. They realized that, on any smooth complex algebraic variety X, there is a natural abelian category inside the nonabelian constructible derived category. It is just the essential image, under the Riemann–Hilbert correspondence, of the category of regular holonomic $\mathscr{D}$-modules $\mathscr{M}$, viewed as complexes concentrated in degree 0. Can this unexpected abelian subcategory be characterized topologically?

That fall and winter, Deligne in Paris and Beilinson and Bernstein in Moscow independently proved what they had conjectured together; then they combined their work and published it in a joint Astérisque monograph. First of all, they proved this theorem: *The* *essential image of the Riemann–Hilbert correspondence consists of the bounded complexes* **S** *with constructible cohomology sheaves* $\mathbf{H}^i(\mathbf{S})$ *satisfying the following two dual conditions*:

(i) $\mathbf{H}^i(\mathbf{S}) = 0$ *for* $i < 0$ *and* $\operatorname{codim}(\operatorname{Supp}(\mathbf{H}^i(\mathbf{S}))) \geq i$ *for* $i \geq 0$,

($i^\vee$) $\mathbf{H}^i(\mathbf{S}^\vee) = 0$ *for* $i < 0$ *and* $\operatorname{codim}(\operatorname{Supp}(\mathbf{H}^i(\mathbf{S}^\vee))) \geq i$ *for* $i \geq 0$.

Conditions (i) and ($i^\vee$), in fact, define a full abelian subcategory also if X is an algebraic variety in arbitrary characteristic p with the étale topology. The conditions can be modified using an arbitrary perversity so that they still yield a full abelian subcategory; the original conditions are recovered with the middle perversity. Moreover, unlike arbitrary complexes in the derived category, those **S** that satisfy the modified conditions can be patched together from local data like sheaves. Because of all those marvelous properties, everyone calls these special complexes **S** (or sometimes, their shifts by $d := \dim_{\mathbf{C}}(X)$) *perverse sheaves.* Of course, they are complexes in a derived category, not sheaves. And, they are well behaved, not perverse. Nevertheless, the name has stuck.

Beilinson, Bernstein, and Deligne also proved the following two theorems: (1) *The abelian category of perverse sheaves is Noetherian and Artinian, every object has finite length.* (2) *Let* V *be a smooth, irreducible subvariety of codimension* c *of* X, *and* **L** *a locally constant sheaf of vector spaces on* V. *Then* (a) *there is a unique perverse sheaf* **S** *whose restriction to* V *is* $\mathbf{L}[-c]$; (b) *if* **L** *is the constant sheaf* $\mathbf{C}_V$, *then* **S** *is equal to the shifted intersection homology complex* $\mathbf{IC}^\bullet(\overline{V})[-c]$, *where* $\overline{V}$ *is the closure of* V; *in general,* **S** *can be constructed from* **L** *by the same process of repeated pushforth and truncation*; (c) *if* **L** *is an irreducible locally constant sheaf, then* **S** *is a simple perverse sheaf. Conversely, every simple perverse sheaf has this form.*

The perverse sheaf **S** of (2) is denoted $\mathbf{IC}^\bullet(\overline{V}, \mathbf{L})[-c]$ and called the *DGM extension*, or Deligne-Goresky-MacPherson extension, of **L**. It is also called the *twisted intersection cohomology complex with coefficients in* **L**. Thus, the family of intersection cohomology complexes was enlarged through twisting and then became merely the family of simple objects in the remarkable new abelian category of perverse sheaves.

8. Purity and decomposition

About July 1980, Gabber settled the matter of "purity" that Deligne posed in his letter to Kazhdan and Lusztig. In fact, he proved more: *Any DGM extension* $\mathbf{IC}^{\cdot}(\overline{V}, \mathbf{L})[-c]$ *is "pure of weight* $-c$*" in the more sophisticated sense of Deligne's second great paper on the Weil conjectures.* Thus, in particular, there are unexpectedly many pure complexes to which to apply Deligne's theory.

The Weil–E. Artin–Riemann hypothesis and the hard Lefschetz theorem are, as already noted, two major corollaries of the purity theorem. However, the single most important corollary, is doubtless, the following "decomposition theorem": *If* $f: X \to Y$ *is a proper map of varieties, then* $\mathbf{R}f_*\mathbf{IC}^{\cdot}(X)$ *is a direct sum of shifts of DGM extensions* $\mathbf{IC}^{\cdot}(\overline{V}_i, \mathbf{L}_i)[-e_i]$, *where* e_i *is not necessarily the codimension of* V_i. These three corollaries hold for varieties defined over an algebraically closed field; for the Riemann hypothesis, it must be the algebraic closure of a finite field, but for the Lefschetz theorem and the decomposition theorem, it may be arbitrary, even the field of complex numbers $\mathbf{C}$!

The decomposition theorem was conjectured in the spring of 1980 by Sergei Gelfand and MacPherson, then proved that fall by Gabber and Deligne and independently by Beilinson and Bernstein. Over $\mathbf{C}$, an analytic proof of it and of the hard Lefschetz theorem, based on a theory of *polarizable Hodge modules* analogous to the theory of pure perverse sheaves, was given several years later by Morihiko Saito.

Sergei Gelfand and MacPherson showed that the decomposition theorem yields Kazhdan and Lusztig's main theorem, stated at the very end of §5 above, which relates their polynomials to the intersection homology groups of the Schubert varieties. The proof involves a lovely interpretation of the Hecke algebra as an algebra of correspondences. Moreover, given the decomposition theorem over $\mathbf{C}$, the proof involves no reduction to positive characteristic; thus, by using polarizable Hodge modules, the Weil conjectures (that is, purity) may be eliminated from the proof of the Kazhdan-Lusztig conjecture.

9. The book under review

In the preface, Kirwan sets the tone and agenda:

> These notes are based on a course for graduate students entitled 'A beginner's guide to intersection homology theory' given in Oxford in 1987. The course was intended to be accessible to first year graduate students and to mathematicians from different areas of mathematics. The aim was to give some idea of the power, usefulness and beauty of

> intersection homology theory while only assuming fairly basic mathematical knowledge. To succeed at all in this it was necessary to give at most briefly sketched proofs of the important theorems and to concentrate on explaining the main ideas and definitions. The result is that these notes do not constitute in any sense an introductory textbook on intersection homology. Rather they are intended to be a piece of propaganda on its behalf. The hope is that mathematicians of very varied backgrounds with interests in singular spaces should find the notes readable and should be stimulated to learn in greater depth about intersection homology and use it in their work. ...
>
> The goal I had in mind was to explain enough of the theory of intersection homology to be able to give a sketch ... of the proof of the Kazhdan-Lusztig conjecture ...
>
> This goal influenced the structure of the second half of the course and thus the lecture notes. The first half consists of an elementary introduction to intersection homology theory ... .

Kirwan, who professes in the preface to be "an enthusiast for intersection homology ... although by no means an expert on the subject" does an admirable job in carrying out that agenda, particularly on the more elementary material. The book is so readable that one wishes it would go on longer in the same spirit.

The discussion of l-adic cohomology is flawed, however. The material on pp. 98 and 104 might give a false impression about the definition of the group $H^i(Y, \mathbf{Q}_l)$. It is not simply the cohomology group of the constant sheaf on the field $\mathbf{Q}_l$ of l-adic numbers in the étale topology; rather, it is the tensored inverse limit of the cohomology groups with torsion coefficients,

$$H^i(Y, \mathbf{Q}_l) := (\varprojlim H^i_{\text{ét}}(Y, \mathbf{Z}/l\mathbf{Z})) \otimes \mathbf{Q}_l.$$

The distinction is crucial. Also, the material on pp. 106–107 might give the impression that the Frobenius map is an isomorphism of varieties, whereas it is only an isomorphism of the étale topologies.

One unfortunate omission is mention of Zucker's celebrated 1980 conjecture.

In sum, the book may be highly recommended (with the caveat above) to beginners who wish a bird's-eye view of this broad and beautiful, but sometimes deep and sophisticated theory.

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Unit groups of classical rings, by Gregory Karpilovsky, Clarendon Press, Oxford, 370 pp., \$98.00. ISBN 0-19-853557-0

Call a ring unitary if it has an identity element under multiplication. If R is a unitary ring, then there are several groups and monoids that are naturally associated with R. Among these are the additive group $(R, +)$ of R (that is, the group on the set R with operation the operation of addition defined on the ring R), the multiplicative monoid $(R, \cdot)$ of R, and the multiplicative group $U(R)$ of units of R. (A unit of R is an element that has a multiplicative inverse in R; for example 1 and -1 are the units of the ring of integers.) Ring theorists have long been interested in the interplay and relations that exist between the algebraic structures R, $(R, +)$, $(R, \cdot)$ and $U(R)$. Clearly R nominally determines the other three structures. What about the converse? To what extent do one or more of the structures $(R, +)$, $(R, \cdot)$ and $U(R)$ determine R? A different kind of question concerns realization: for example, given an abelian group G and a group H, can G and H be realized as the additive and unit groups, respectively, of a unitary ring R, and if so, how many realizations are there, to within isomorphism? To illustrate this last question, suppose $G = Z$, the infinite cyclic group. If G is the additive group of a unitary ring R, and if g is a generator for G, then the multiplication on R is completely determined by the integer k, where $g^2 = kg$; moreover, $k = \pm 1$ since R is unitary. Since $(-g)^2 = (-k)(-g)$, where $-g$ is also a generator for G, it follows that R is isomorphic to the ring of integers, so H must be cyclic of order two in order for the pair (G, H) to be realizable. In a similar vein, Chapter 6 of the book under review determines the unitary rings R for which $U(R)$ is cyclic. Natural variants on these themes arise if one restricts to rings or groups that satisfy a given condition E. For example, early work by Fuchs, Szele