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CLAUDE CHEVALLEY (1909–1984)

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Both parents of Claude Chevalley came from Protestant families. His mother was the daughter of a Calvinist minister from southern France, who had a distinguished career in the Protestant hierarchy, culminating in the deanship of the Protestant Faculty of Theology in Paris. Chevalley's father was the son of a Swiss watchmaker who had settled in western France. He held several teaching positions in secondary schools before entering the diplomatic service; when Chevalley was born, his father was the French consul in Johannesburg.

Chevalley's intellectual gifts were soon apparent, and allowed him to enter the École Normale Supérieure in Paris at the early age of 17. There he met J. Herbrand, who was one year older, and with whom he struck a deep friendship, unfortunately broken when Herbrand died in a mountain climbing accident in 1931. Both starting doing research while still students at the École Normale; they were interested in topics that were taught nowhere in France at the time, such as mathematical logic, number theory and algebra; after their graduation, with the help of research grants (very scanty at that time), they were able to visit German universities where these fields were being actively developed. The research they did there was strongly influenced by the work of E. Noether, E. Artin and H. Hasse; it was chiefly concerned with the theory of algebraic numbers, and was highly valued by the German school.

The main contributions of Chevalley during the years 1930–1940 were focused on both local and global class field theory. What is now called "global" class field theory deals with abelian extensions of number fields and has its origin in the early results of Kronecker, H. Weber, and Hilbert at the end of the 19th century. Inspired by the special cases Kronecker and Weber had treated, Hilbert had formulated general conjectures; these (later generalized by Takagi) established a one-to-one natural correspondence between the abelian extensions of a number field K and certain classes of ideals in K, and also described how a prime ideal in K splits in an abelian extension of K, in terms of that correspondence. Hilbert's conjectures were proved between 1910 and 1923 by Furtwängler and Takagi, whose results were completed in 1927 by E. Artin's famous "reciprocity law" which exhibited an explicit isomorphism between the Galois group of an abelian extension E of K and a quotient group of the group of ideal classes of K, in the construction of which there entered the norms in K of the prime ideals of E.

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The proofs of these theorems were extremely complicated; Herbrand and Chevalley were able to substantially simplify them, and these improvements were incorporated in Chevalley's thesis of 1933 **[4]**. It was also in his thesis that he laid the foundations of an autonomous local class field theory, which until then had been derived by Hasse from the global theory, by localizing the number field at a prime ideal $\mathfrak{p}$. Chevalley showed that it is possible to start directly from a $\mathfrak{p}$-adic field and to consider its abelian extensions. The main theorems are then much simpler than in the global theory; the one-to-one correspondence is between abelian extensions E of a $\mathfrak{p}$-adic field K and the subgroups of finite index in the multiplicative group $K^\times$ of nonzero elements of K; the Galois group of E over K is then naturally isomorphic to the quotient of $K^\times$ by the group of norms over K of the elements of the multiplicative group $E^\times$.

In the following years, Chevalley's efforts were directed towards the elimination from class field theory of the tools involving analytic number theory applied to zeta functions, which had been essential in the proofs of his predecessors. He also wanted to generalize class field theory to number fields which are extensions of infinite degree of the rational field. He succeeded in both endeavors by the introduction of the new concept of *idèle* **[11]**, which has become fundamental in algebraic number theory, where it is the first explicit instance of "passage from local to global." A number field K of finite degree is embedded into the product $\prod_v K_v$ of all completions K_v of K at the places v of K, *both finite and infinite*; this is a definite improvement on earlier similar ideas of Prüfer and von Neumann, who had only embedded K into the product over the *finite* places. In fact, the product $\prod_v K_v$ is too big; Chevalley restricted it to the set of elements (ξ_v) such that $\xi_v \neq 0$ for all v, and for finite places v, ξ_v is a unit in the ring of integers of K_v, except for a finite number of places. These elements are the idèles of K; they form a multiplicative group J_K; it contains the multiplicative group $K^\times$ as a subgroup, when one identifies an element $\alpha \in K^\times$ with the idèle whose components ξ_v are all equal to α. If E is an abelian extension of K, there is a norm homomorphism $J_E \to J_K$: if (ζ_w) is an idèle in J_E, the corresponding idèle (ξ_v) in J_K is obtained by taking for each place v all places w of E above v, and the norms over K_v of the elements ζ_w; ξ_v is the product of all these norms for the places w above v; the image of J_E in J_K is a subgroup written $N(J_E/J_K)$. The main theorem of global class field theory, expressed in terms of idèles, is that the Galois group $\mathrm{Gal}(E/K)$ of an abelian extension is isomorphic to

$$J_K/K^\times N(J_E/J_K)$$

and there is a natural isomorphism of that group onto $\mathrm{Gal}(E/K)$, expressed in terms of Artin symbols. In 1940, Chevalley was able to obtain that result directly from the local theory, using neither Takagi's results nor analytic number theory **[13]**. He made great use of topological notions in the group J_K, but the topology he introduced was not the best one for the theory, and it was later replaced by a more useful one, for which J_K is a locally compact abelian group, to which the general theory of such groups (in particular Fourier theory) may be applied, with beautiful results.

In 1939, Chevalley had been invited for a semester to the Institute for Advanced Study, and he was in Princeton when the war broke out. The French ambassador suggested that he stay in the USA for the time being, and after the German invasion he could not go home. He was offered a position on the faculty at Princeton University, and taught there until 1948. After that date, he became a Professor at Columbia University, where he stayed until 1955; during the year 1953–54 he visited Japan on a Fulbright scholarship, and gave several series of lectures at Japanese universities. His lectures always were at a very high level of rigor and did not attract average students; but several bright graduate students did research work under his guidance, and his encyclopaedic knowledge and original ideas always were very much appreciated by mathematicians who exchanged ideas with him.

After 1940, Chevalley turned his attention to mathematical fields in which he had not previously done much work, Lie groups and algebraic geometry. In 1941, he became interested in algebraic geometry over an arbitrary field, which A. Weil, who had been able to escape from France to the USA, was at that time building up on new bases. Chevalley proved several key properties of the local rings of an algebraic variety **[16]**, and constructed an original theory of intersections, which used methods different from those of Weil, and could be applied to algebroid varieties as well **[21]**. For more than ten years he actively remained interested in the foundations of algebraic geometry, both for their own sake—they were the essential topic of his joint seminar with H. Cartan* **[51]**—and in view of his work on algebraic groups. All his life, in fact, and besides his deepest and highly technical mathematical investigations, Chevalley was passionately concerned with problems of foundations, a concern inseparable from his great interest in philosophy.

But Chevalley's most important contribution to mathematics is certainly his work on group theory. Here, one must first mention the trilogy: *Theory of Lie groups*, I, **[41]**, *Théorie des groupes de Lie*, II **[43]** and III **[44]**. The titles suggest that they are three successive parts of a same opus. In fact, the three books are very different in conception and scope (not to mention the language and the publisher!). Part one was the first systematic exposition of the foundations of Lie group theory consistently adopting the global viewpoint, based on the notion of analytic manifold. This book remained the basic reference on Lie groups for at least two decades.

But in the late forties, Chevalley became more and more interested in the purely algebraic (so-to-speak still more global) aspects of the theory. This shift of focus is already apparent in I, the last chapter of which is very algebraic in nature; in particular it contains Chevalley's well-known result that "compact Lie groups are algebraic." The evolution is complete in part II, which takes one over the whole theory again (it is essentially independent from I) but in the framework of algebraic Lie groups and Lie algebras. By working over an arbitrary field of characteristic zero, the author precludes any use of analytic methods. An important tool created on that occasion (in fact, already introduced in an earlier paper **[17]**) is that of "replica of a matrix M": by

*Here, the French word "schéma" appears for the first time in algebraic geometry. The influence of that seminar on Grothendieck's ideas is worth mentioning.

definition, these are all elements of the smallest algebraic Lie algebra (i.e. the Lie algebra of a linear algebraic group) containing M. As an application of that notion, Chevalley proves the following criterion for a subalgebra $\mathfrak{g}$ of $\mathfrak{gl}_n$ to be algebraic: this is the case if and only if $\mathfrak{g}$ contains all replicas of its elements. Part III is a somewhat more standard exposition of fundamental results in the theory of Lie algebras.

Of greatest significance and with major impact was Chevalley's most seminal paper entitled *Sur certains groupes simples*, published in 1955 in the Tôhoku Mathematical Journal [**36**]. Since Killing and E. Cartan it was known that the only existing simple complex analytic groups were the so-called classical groups and the five exceptional groups G_2, F_4, E_6, E_7, E_8. In the mid-fifties, classical groups over arbitrary fields had already been defined and extensively studied; algebraic analogues of some of the exceptional groups had been known for quite a while, and several mathematicians, including Chevalley himself [**26, 29, 30**], were pursuing investigations of the individual exceptional groups, in order to better understand them and, for some people, to define algebraic analogues for all of them. This last goal was attained in all cases at once in the "Tôhoku paper." The importance and originality of that paper lies less in the *new simple groups* it was exhibiting (in fact, all "Chevalley groups" except those of type E_8 had already been or were on the verge of being constructed by *ad hoc* methods), than in the unified treatment of the groups involved. This inaugurated a completely *new theory* and revealed important structural features of a general nature. It opened the way to further progress both in the theory of algebraic simple groups and that of finite simple groups, whose classification could not have been contemplated without the understanding of the "Chevalley groups" provided by the "Tôhoku" approach.

The crucial discovery of Chevalley in that paper is the fact that a complex simple Lie group, considered as a linear group via the adjoint representation, can be described by means of a generating system of "one-parameter" (additive or multiplicative) subgroups, given by polynomial formulas with *integral coefficients*, this being of course true only with respect to cleverly chosen bases of the Lie algebra, now called *Chevalley bases*. Once the group is given such a description, where all coefficients in the formulas are rational integers, it is a simple matter to define "analogues" of that group over an arbitrary field K: indeed, it suffices to let the variables run over K instead of $\mathbf{C}$, the field of complex numbers.

After Grothendieck, we now recognize in that process a "change of base," and we suspect that what Chevalley had actually done in the Tôhoku paper was to associate to every simple complex Lie group G a simple group scheme $\mathcal{G}$ over $\mathbf{Z}$ such that G is the group $\mathcal{G}(\mathbf{C})$ of complex points of $\mathcal{G}$, the Chevalley groups being the corresponding $\mathcal{G}(K)$. The conceptual step was indeed accomplished by Chevalley himself (this time, not only for the adjoint groups but also for their various coverings) in a lecture at the Séminaire Bourbaki in 1961 [**39**].

In 1955, Chevalley returned to France, where he became a professor in the University of Paris; he taught there until his retirement in 1978, and started a seminar, chiefly devoted to algebraic geometry and the theory of groups. It is there that in 1956–58 he lectured on what many consider his masterpiece, at least as important and of even more prowess than the "Tôhoku paper": the determination of all semisimple groups over an arbitrary algebraically closed field. The mimeographed notes of that seminar were nicknamed "The Bible" by the specialists; curiously enough, those notes of fundamental importance, still currently used by many mathematicians, never appeared in print **[49]**.

The classification of *complex* semisimple groups, due to W. Killing and E. Cartan, was more than half a century old. It was a relatively simple matter to transfer the theory from the complex field to an arbitrary algebraically closed field of characteristic zero: this was in fact essentially done in Chevalley's books already mentioned. But the methods of Killing and Cartan belonged to what is now called the theory of Lie algebras, and there was no hope to extend them to the case of finite characteristic, for at least two reasons: Chevalley himself had observed that in characteristic $p > 0$, an algebraic group is no longer determined locally by its Lie algebra (in heuristic terms, the latter only determines an "infinitesimal neighborhood of order p" of the identity in the group). Moreover it was known that the classification of simple Lie algebras in characteristic p—a classification that is not yet completed to this day—was much more complicated and led to a longer enumeration than in characteristic zero. For those reasons, it was widely believed in the early fifties that the realm of algebraic simple groups was much richer and intricate in finite characteristic than in characteristic zero.

When Chevalley was able to solve the problem, it therefore came as a great surprise that, although the methods were completely different from those of Killing and Cartan, the final result was independent of the characteristic: existence of four infinite classes of "classical groups" and of five "exceptional groups"; and a like number of "central isogeny classes" (groups that are, in a suitably defined way, "locally isomorphic" for each group. Along the way, Chevalley was also able to extend E. Cartan's theory of irreducible linear representations of semisimple groups. Here again, the result is "characteristic-free," to some extent at least: irreducible representations are classified by the same "highest weights" as in characteristic zero; however, Weyl's dimension formula is no longer valid and one does not have complete reducibility for arbitrary representations.

Lie algebra techniques being no longer available (or rather, being useless), Chevalley had to devise purely group-theoretical and algebro-geometric methods. This was made possible by A. Borel's work on algebraic linear groups (Annals of Math., 1956) of which Chevalley's seminar is in fact a continuation. Borel's fundamental discovery was that some essential structural properties of complex Lie groups, which had been established by transcendental methods or using the classification, could be proved very simply by means of purely algebro-geometric arguments which worked equally well over arbitrary algebraically closed fields; thus, he could extend to those fields such properties as the conjugacy (in a connected algebraic group) of all maximal connected solvable subgroups—now called Borel subgroups—and the conjugacy of all maximal tori. An essential result in Borel's theory is that a coset space G/P of a connected algebraic linear group G by a closed subgroup P is a projective variety if and only if P contains a Borel subgroup. In particular, to every such group G there is associated in a most canonical fashion one especially remarkable projective variety, namely G/B for B a Borel subgroup. Maximal tori, Borel subgroups, and the variety G/B are the major ingredients of Chevalley's proof, which is much longer and more involved than the proof of Killing and Cartan, and is a monument of technical skill and ingenuity.

Although by far the most important single piece of work of Chevalley on the subject, the "Séminaire" does not exhaust his contributions to the theory of algebraic groups. Especially worth mentioning are an unpublished paper describing the Chow ring of G/B, and the important theorem according to which every algebraic group G is in a unique way an extension of an abelian variety by a linear algebraic group (i.e., G has a unique closed maximal linear subgroup L which is normal, and G/L is an abelian variety).

In the sixties, Chevalley turned his attention to finite group theory, which he made the main subject of his teaching and seminars. In that way, he created an active school responsible for important new developments in Brauer's theory of blocks and modular representations.

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