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THE CUSPED HYPERBOLIC 3-ORBIFOLD OF MINIMUM VOLUME

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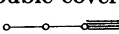
An orbifold is a space locally modelled on $\mathbf{R}^n$ modulo a finite group action. We will restrict our attention to complete orientable hyperbolic 3-orbifolds Q; thus, we can think of Q as H^3/Γ, where Γ is a discrete subgroup of $\mathrm{Isom}_+(H^3)$, the orientation-preserving isometries of hyperbolic 3-space. An orientable hyperbolic 3-manifold corresponds to a discrete, *torsion-free* subgroup of $\mathrm{Isom}_+(H^3)$. We will work in the upper-half-space model H^3 of hyperbolic 3-space, in which case $PGL(2,\mathbf{C})$ acts as isometries on H^3 by extending the action of $PGL(2,\mathbf{C})$ on the Riemann sphere (boundary of H^3) to H^3. If the discrete group Γ corresponding to Q has parabolic elements, then Q is said to be cusped. (For more details on this paragraph see [**T**, Chapter 13].)

Unless otherwise stated, we will assume all manifolds and orbifolds are orientable. Mostow's theorem implies that a complete, hyperbolic structure of finite volume on a 3-orbifold is unique. Consequently, hyperbolic volume is a topological invariant for orbifolds admitting such structures. Jørgensen and Thurston proved (see [**T**, §6.6]) that the set of volumes of complete hyperbolic 3-manifolds is well-ordered and of order type ω^ω. In particular, there is a complete hyperbolic 3-manifold of minimum volume V_1 among all complete hyperbolic 3-manifolds and a cusped hyperbolic 3-manifold of minimum volume V_ω. Further, all volumes of closed manifolds are isolated, while volumes of cusped manifolds are limits from below (thus the notation V_ω).

Modifying the proofs in the Jørgensen-Thurston theory yields similar results for complete hyperbolic 3-orbifolds (but see the remark at the end of this paper). In particular, there is a hyperbolic 3-orbifold of minimum volume, and a cusped hyperbolic 3-orbifold of minimum volume. We prove

THEOREM. *Let* $Q_1 = H^3/\Gamma_1$ *where* $\Gamma_1 = PGL(2, \mathcal{O}_3)$ *and* $\mathcal{O}_3$ *=ring of integers in* $Q(\sqrt{-3})$. *The orbifold* Q_1 *has minimum volume among all orientable cusped hyperbolic 3-orbifolds.*

Note. Q_1 is the orientable double-cover of the (nonorientable) tetrahedral orbifold with Coxeter diagram  (see [**T**, Theorem 13.5.4] and [**H**, §1]). This tetrahedral orbifold has fundamental domain $1/24$ of the ideal regular hyperbolic tetrahedron (use the symmetries). In particular, Q_1 has a cusp and its volume is $1/12$ the volume of the ideal regular tetrahedron T, i.e. $\mathrm{vol}(Q_1) = V/12 \approx 0.0846$, where $V = \mathrm{vol}(T)$.

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PROOF (OF THEOREM). In Parts I and II of the proof we will get a lower bound for the volume of H^3/Γ for arbitrary cusped discrete Γ.

PART I: VOLUME CONTRIBUTIONS OF CUSPED NEIGHBORHOODS IN H^3/Γ.

MANIFOLD CASE (i.e., Γ such that H^3/Γ is a manifold with a cusp): We can assume (using a suitable conjugation) that the cusp corresponds to the point at ∞ in H^3, and that the parabolic transformation $z \mapsto z+1$ is the "shortest" element in Γ_∞, the stabilizer of ∞ in Γ (Γ_∞ has no hyperbolic elements; see [**Be**, Theorem 5.1.2]). Construct the horoball C_∞, centered at ∞, for which Γ_∞ has minimum translation length one (in the Euclidean metric) on the horosphere boundary of C_∞. Our set-up has been rigged so that $C_\infty = \{(x, y, t) : t \geq 1\}$. Construct such "length one" cusp neighborhoods at all parabolic fixed points (for some element of Γ). It is a standard fact (see [**Be**, Theorem 5.4.4]) that all such cusp neighborhoods are disjoint. Thus C_∞/Γ_∞ is an embedded "cusp neighborhood" in $M = H^3/\Gamma$.

What is the volume of C_∞/Γ_∞? If $z \mapsto z+1$ is the "shortest element" in Γ_∞, then any other element $z \mapsto z+w$ in Γ_∞ must have $|w| \geq 1$ and $|\mathrm{Im}(w)| \geq \sqrt{3}/2$. Thus, we can compute $\mathrm{vol}(C_\infty/\Gamma_\infty) \geq \sqrt{3}/4$ (see [**M1**, §5]).

ORBIFOLD CASE. The only additional complication from the manifold case is that Γ_∞ may include elliptic elements. If so, then the elliptic and parabolic elements comprising Γ_∞ act as rigid motions on the (Euclidean) horosphere at height 1 in H^3. Thus, we need only study the oriented wall-paper groups to understand the effect of the elliptic elements on the volume estimate for C_∞/Γ_∞. There are 5 such wall-paper groups, and the worst case reduces volume by a factor of 6.

The cusp neighborhoods contribute at least $\sqrt{3}/24$ *to the volume of a complete orientable cusped hyperbolic* 3*-orbifold.*

PART II: VOLUME CONTRIBUTIONS OUTSIDE THE CUSP NEIGHBORHOODS. By Part I, we have some control over the size of a cusped neighborhood. However, this cusp neighborhood is only a portion of the fundamental domain for Γ. Can we gain some control over the size of the fundamental domain outside of the cusp neighborhood? Yes, by sphere-packing. First, we fix a particular fundamental domain D for Γ: Let $D_\infty = \{p \in H^3 : p$ is closer to C_∞ than to any conjugate (under Γ) of $C_\infty\}$. Then we take D to be a fundamental domain for the action of Γ_∞ on D_∞.

Next, consider 4 horospheres in H^3, each touching all the others. Their centers (points of tangency with ∂H^3) will determine an ideal regular tetrahedron T. Let B be the union of the 4 horoballs bounded by the 4 horospheres. Böröczky's theorem (see [**B**, Theorem 4]) says that this is, in some sense, the densest packing of horospheres in hyperbolic 3-space. In terms of C_∞ and D, Böröczky's theorem implies that $\mathrm{vol}(C_\infty \cap D)/\mathrm{vol}(D) \leq \mathrm{vol}(B \cap T)/\mathrm{vol}(T) = 4(\sqrt{3}/8)/V = \sqrt{3}/2V$ (for more details, see [**M2**]).

Thus, $\mathrm{vol}(H^3/\Gamma) = \mathrm{vol}(D) \geq \mathrm{vol}(C_\infty \cap D)/(\sqrt{3}/2V) \geq (\sqrt{3}/24)(2V/\sqrt{3}) = V/12$.

PART III: SUMMARY. As mentioned above, the orbifold Q_1 has a cusp and has volume $V/12$. Parts I and II tell us that all cusped orbifolds have volume at least $V/12$. Thus Q_1 realizes the minimum volume and it is $V/12 \approx 0.0846$. □

REMARK. There are cusped orbifolds on which Dehn surgery cannot be performed. Consequently, unlike the manifold case, there are cusped hyperbolic 3-orbifolds whose volumes are isolated—Q_1 is such an orbifold. The question of finding "the least limiting orbifold" remains open.

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