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THE FIRST EIGENVALUE IN A TOWER OF COVERINGS

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Let M be a compact Riemannian manifold, and let $\{M_i\}$ be a family of finite Riemannian covering spaces of M. Let $\lambda_1(M_i)$ be the first eigenvalue of the Laplacian on M_i. λ_1 is given by the variational formula

$$\lambda_1(M_i) = \inf_f \frac{\int_{M_i} \|df\|^2}{\int_{M_i} |f|^2},$$

where f ranges over functions satisfying $\int_{M_i} f = 0$.

In this note we announce results on the following problem: When is there a sequence of i's where $\lambda_1(M_i)$ is bounded from below as $i \to \infty$? Our approach to this problem is of a piece with our approach to studying eigenvalue problems related to λ_0 in [**1** and **3**]. Namely, we reduce the eigenvalue problem to a combinatorial problem built out of the fundamental group.

To state the combinatorial problem, let us pick generators $g_1, \ldots, g_k$ for $\pi_1(M)$. Consider, for each i, the finite graph Γ_i described as follows: the vertices of Γ_i are the cosets $\pi_1(M)/\pi_1(M_i)$. Two vertices are joined by an edge if they differ by left multiplication by one of the g_i's.

For each i, let $h_i = h(\Gamma_i)$ denote the following number: Let $E = \{E_j\}$ be a collection of edges of Γ_i such that $\Gamma_i - E$ disconnects into two pieces,

$$\Gamma_i - E = A \cup B.$$

Denote by $\#(E)$ the number of elements of E, and $\#(A)$ (resp. $\#(B)$) the number of vertices in A (resp. B). Then

$$h_i = \inf_E \frac{\#(E)}{\min(\#(A), \#(B))}.$$

h_i is of course the combinatorial analogue of Cheeger's isoperimetric constant [**11**].

THEOREM 1. *There is a positive constant C_1 such that $\lambda_1(M_i) \geq C_1$ for all i if and only if there is a positive constant C_2 such that $h_i \geq C_2$ for all i.*

The general question of describing manifolds M for which $\lambda_1(M_i)$ is uniformly bounded away from 0 was raised recently by Sunada [**10**]. Letting $\{M_i\}$ range over all coverings of M, Theorem 1 provides a combinatorial answer to his question. Of course, the problem remains of finding good criteria to check when this combinatorial problem is solved. We comment on some examples at the end of this paper.

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Note that the theorem shows that the combinatorial condition does not depend on the choice of generators of $\pi_1(M)$. Using this observation, we can show

THEOREM 2. *Suppose that there is a surjective homomorphism $\pi_1(M) \to \pi_1(N)$. If N has coverings N_i with $\lambda_1(N_i) \to 0$, then the same is true of M. If N has coverings N_i with $\lambda_1(N_i) \geq C$ for some positive C, then the same is true of M.*

COROLLARY 3. *Suppose $\pi_1(M)$ has a homomorphism onto a group Γ which is infinite, amenable, and residually finite. Then M has coverings M_i with $\lambda_1(M_i) \to 0$.*

PROOF. The condition that Γ is amenable means that there are subsets $\{V_i\}$ of the graph of Γ with $\#(\partial V_i)/\#(V_i) \to 0$ as $i \to \infty$. Residual finiteness and the fact that Γ is infinite then implies that there are subgroups Γ_i of finite index in Γ such that V_i maps injectively to Γ/Γ_i, and $\#(V_i) \leq \frac{1}{2}\#(\Gamma/\Gamma_i)$.

Noting that $\mathbf{Z}$ is a group which is well known to be amenable and residually finite, we might regard Corollary 3 as a generalization of a result of B. Randol [**7**].

A result essentially like Corollary 3 was given by T. Sunada in his beautiful paper [**10**].

Using techniques from [**2** and **4**], we may weaken the hypothesis that M be compact in Theorem 1. It suffices to assume only that M is of finite topological type, finite volume, and satisfies an "isoperimetric condition at infinity." In particular, Theorem 1 applies to Riemann surfaces of finite type, endowed with a complete metric of finite area and constant curvature.

The following theorem was proved by A. Selberg [**9**], see also Sarnak [**8**].

THEOREM (SELBERG). *Let $\Gamma = \mathrm{PSL}(2, \mathbf{Z})$, and for each n let Γ_n be the congruence subgroup*

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2, \mathbf{Z}) \colon \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{n} \right\}.$$

Then there is a constant C such that $\lambda_1(\mathbf{H}^2/\Gamma_n) \geq C$ for all n.

Selberg gives a value of $3/16$ for C. At present we do not have a purely combinatorial proof, via Theorem 1, of Selberg's result (with, presumably, a smaller constant). We note, however, the following consequence, settling a question raised by P. Buser [**6**].

THEOREM 4. *Let S be a hyperbolic Riemann surface of finite type. Then there is a positive constant C and infinitely many coverings S_i of S such that $\lambda_1(S_i) \geq C$.*

PROOF. $\pi_1(S)$ has a homomorphism onto a free group on two generators, and hence onto $\mathrm{PSL}(2, \mathbf{Z})$. Theorem 2 and Selberg's theorem then imply Theorem 4.

Note that, somewhat oddly, the coverings S_i constructed from Theorem 4 have plenty of short geodesics, namely those in the kernel of $\pi_1(S) \to \mathrm{PSL}(2,\mathbf{Z})$.

The proof of Theorem 1 follows closely the proof of the main theorem of [**3**]. There are two steps.

Step 1. If $h_i \to 0$ for some sequence of i, then one constructs test functions f_i on M_i which are essentially 1 on fundamental domains corresponding to vertices in A, $-d$ on fundamental domains corresponding to B, where $d \leq 1$ is chosen to make $\int_{M_i} f_i = 0$, and which taper off to 0 in some standard way where domains lying in A meet domains lying in B, corresponding to edges lying in E. It is easily seen that the Rayleigh quotients of f_i tend to 0.

Step 2. Now assume that $\lambda_1(M_i) \to 0$. According to Cheeger's inequality [**11**], there exist hypersurfaces S_i on M_i dividing M_i into two pieces $M_i - S_i = M_{i,1} \cup M_{i,2}$, such that

$$\frac{\operatorname{area}(S_i)}{\min(\operatorname{vol}(M_{i,1}), \operatorname{vol}(M_{i,2}))} \to 0 \quad \text{as } i \to \infty.$$

We now try to choose hypersurfaces S_i which minimize the isoperimetric constant on M_i. As is shown in [**3**], the minimum will be achieved for some integral current T_i, which is of constant mean curvature. According to an argument of Buser in [**5**], the mean curvature of T_i will be bounded independent of i.

Arguments similar to those in [**3**] now show that one may essentially take A and B to be those fundamental domains lying on $M_{i,1}$ and $M_{i,2}$ respectively, completing the proof of Theorem 1.

We remark that there are examples of manifolds M with infinitely many coverings such that there is a positive constant C with $\lambda_i(M_i) \geq C$ for *all* coverings M_i of M. The known examples of these are arithmetic groups which have Kazhdan's Property T. Any such group could have been taken instead of $\mathrm{PSL}(2,\mathbf{Z})$ in Theorem 4.

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