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ARITHMETIC CHARACTERIZATIONS OF SIDON SETS

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ABSTRACT. Let $\hat{G}$ be any discrete Abelian group. We give several arithmetic characterizations of Sidon sets in $\hat{G}$. In particular, we show that a set Λ is a Sidon set iff there is a number $\delta > 0$ such that any finite subset A of Λ contains a subset $B \subset A$ with $|B| \geq \delta|A|$ which is quasi-independent, i.e. such that the only relation of the form $\sum_{\lambda \in B} \epsilon_\lambda \lambda = 0$, with ϵ_λ equal to ± 1 or 0, is the trivial one.

Let G be a compact Abelian group and let $\hat{G}$ be the dual group. For any f in $L_2(G)$, we denote by $\hat{f}$ the Fourier transform of f. A subset Λ of $\hat{G}$ is called a Sidon set if there is a constant K with the following property: all the trigonometric polynomials f, such that $\hat{f}$ is supported by Λ, satisfy

$$\sum |\hat{f}(\gamma)| \leq K \|f\|_{C(G)}.$$

We will denote by $S(\Lambda)$ the smallest constant K with this property. In the theory of Sidon sets (cf. e.g. [**2**]), there has always been considerable interest in the relations between this analytical definition and the arithmetic properties of the set Λ (in particular, in the case $G = \mathbf{T}$ and $\Lambda \subset \mathbf{Z}$). The aim of this note is to announce several arithmetic characterizations of Sidon sets.

Let us make more precise what we mean here by "arithmetic". We will denote by R_Λ the set of relations (with coefficients in $\{-1, 0, 1\}$) satisfied by Λ, i.e. the set of all finitely supported families $(\epsilon_\lambda)_{\lambda \in \Lambda}$ in $\{-1, 0, 1\}^\Lambda$ such that $\sum_{\lambda \in \Lambda} \epsilon_\lambda \lambda = 0$.

By an "arithmetic" characterization is usually meant one which depends only on the set R_Λ. In [**1**], Drury[1] proved that such a characterization exists, but he could not produce any explicit one. Precisely, he proved the following: let Λ and Λ' be two sets for which there is a bijection $\phi \colon \Lambda' \to \Lambda$ such that the map $\tilde{\phi} \colon R_\Lambda \to R_{\Lambda'}$, defined by $\tilde{\phi}((\epsilon_\lambda)_{\lambda \in \Lambda}) = (\epsilon_{\phi(\lambda')})_{\lambda' \in \Lambda'}$, is also a bijection. Then, Λ is a Sidon set iff the same is true for Λ'. In other words, the property of "being a Sidon set" is determined by R_Λ. We give below several *explicit* arithmetic characterizations, from which the preceding result of Drury follows as a corollary.

To state our results, we will need some notation and terminology. We will denote by I_Λ the set of all finitely supported families $(\epsilon_\lambda)_{\lambda \in \Lambda}$ in $\{-1, 0, 1\}^\Lambda$. For any γ in $\hat{G}$, we will denote by $R(\gamma, \Lambda)$ the number of ways to write γ as

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[1]Drury considers only relations such that moreover $\sum_{\lambda \in \Lambda} \epsilon_\lambda = 0$, but this difference is not significant, since we can replace Λ by the set $\tilde{\Lambda} \subset \hat{G} \times \mathbf{Z}$ defined by $\tilde{\Lambda} = \{(\lambda, 1) | \lambda \in \Lambda\}$.

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a finite sum of the form $\gamma = \sum_{\lambda \in \Lambda} \epsilon_\lambda \lambda$ with $(\epsilon_\lambda)_{\lambda \in \Lambda}$ in I_Λ. For any integer $s \geq 0$, we will denote by $R_s(\gamma, \Lambda)$ the cardinal of the set of those $(\epsilon_\lambda)_{\lambda \in \Lambda}$ in I_Λ such that $\sum |\epsilon_\lambda| = s$ and $\gamma = \sum_{\lambda \in \Lambda} \epsilon_\lambda \lambda$. (Note that we have obviously $R(\gamma, \Lambda) = \sum_{s \geq 0} R_s(\gamma, \Lambda)$.) Let A be a finite subset of Λ. We have the following identity, for all $\delta > 0$.

$$\prod_{\lambda \in A} [1 + \delta(\lambda + \overline{\lambda})] = \sum_{\gamma \in \hat{G}} \gamma \left(\sum_{s \geq 0} \delta^s R_s(\gamma, A) \right). \tag{1}$$

We can now state our main theorem (we will denote by $|A|$ the cardinality of a set A).

THEOREM 1. *Let Λ be a subset of $\hat{G}$ not containing* 0. *The following are equivalent.*

(i) Λ *is a Sidon set.*

(ii) *There is a number $\theta < 1$ such that, for all finite subsets A of Λ, we have*

$$\sum_{s \geq 0} \frac{1}{2^s} R_s(0, A) \leq 2^{\theta |A|}.$$

(iii) *There is a number $\theta < 1$ such that, for all finite subsets A of Λ, we have*

$$\sup_{\gamma \in \hat{G}} R(\gamma, A) \leq 3^{\theta |A|}.$$

(iv) *There is a number $\theta < 1$ such that, for all finite subsets A of Λ, we have*

$$\left\{ \sum_{\gamma \in \hat{G}} R(\gamma, A)^2 \right\}^{1/2} \leq 3^{\theta |A|}.$$

The details of the proof can be found in [**5**]. The equivalence (iii)$\Leftrightarrow$(iv) is easy using the observation that $\sum_{\hat{G}} R(\gamma, A) = 3^{|A|}$. The proof of (i)$\Rightarrow$(ii) uses (1) for $\delta = 1/2$ and the integrability properties of $\sum_{\lambda \in A} \operatorname{Re} \lambda$. The proof relies very much on the previous paper [**4**] and on the following result which is proved in [**5**].

PROPOSITION. *The conditions of Theorem* 1 *are also equivalent to the following.*

(v) *There are numbers $\alpha > 0$ and $\rho < 1$ such that, for any finite subset A of Λ, we have*

$$m\left(\left\{ t \in G \,\middle|\, \inf_{\lambda \in A} \operatorname{Re} \lambda(t) > \rho \right\} \right) \leq 2^{-\alpha |A|}.$$

(vi) *There is a number $\alpha > 0$ such that, for any finite subset A of Λ, we can find points $t_1, \ldots, t_N$ in G, with $N \geq 2^{\alpha |A|}$ such that* $\sup_{\lambda \in A} |\lambda(t_i) - \lambda(t_j)| \geq \alpha$ *for all $i \neq j$.*

The equivalence of (v) and (vi) is formal. The implication (v)$\Rightarrow$(i) yields an affirmative answer to Problem 8.3 in [**4**].

DEFINITION. We will say that a set Λ is a Rider set if there is some $\delta > 0$ such that $\sum_{s \geq 0} \delta^s R_s(0, \Lambda) < \infty$. We will say that Λ is quasi-independent if $R(0, \Lambda) = 1$, or equivalently if $R_s(0, \Lambda) = 0$ for all $s \geq 1$.

Such sets—and finite unions of such sets—are the only known examples of Sidon sets, and the main open problem in this theory is the converse:

PROBLEM. Is every Sidon set a finite union of Rider sets? Is it a finite union of quasi-independent sets?

In the particular case $G = \mathbf{Z}(p)^{\mathbf{N}}$, with p a prime number, a positive answer (as well as a complete arithmetic characterization) was given in [**3**]; very recently, J. Bourgain obtained a positive solution to the above problem, assuming more generally that p is a product of distinct prime numbers (private communication).

Actually, it is rather easy to check (see [**5**]) that any Rider set is a finite union of quasi-independent sets; therefore, the above problem reduces to the second question.

Assume that a set Λ is the union of k quasi-independent sets. In that case, any finite subset A of Λ, of cardinality n, must contain a quasi-independent subset $B \subset A$ with $|B| \geq n/k$. Therefore, if the above problem had a positive solution, any Sidon set should verify the above property for some k. It turns out that this is true.

THEOREM 2. *A subset Λ of $\hat{G}$ is a Sidon set iff*
(vii) *there is an integer k such that any finite subset A of Λ contains a quasi-independent subset $B \subset A$ with $|B| \geq |A|/k$.*

The proof that Sidon sets satisfy (vii) is given in [**5**]. The converse follows from Theorem 2.3 in [**4**], since any quasi-independent set B is a Sidon set with $S(B)$ majorized by some absolute constant. In some sense, Theorem 2 reduces the above problem to a purely combinatorial question: Is every set satisfying (vii) a finite union of quasi-independent sets?

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