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A TORELLI THEOREM FOR SIMPLY CONNECTED ELLIPTIC SURFACES WITH A SECTION AND $p_g \geqslant 2$

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0. The purpose of this note is to announce some recent results on the period mapping for simply connected elliptic surfaces with a section and $p_g \geqslant 2$. Among other things, we are able to prove the period mapping for such surfaces has degree one, that is to say, it is generically injective. In the terminology of **[C, G]**, this is called the weak global Torelli theorem. In order to make a precise statement of our results, we must introduce some notation.

1. Let V be a surface with at most rational double points, $\overline{V} \xrightarrow{\sigma} V$ its minimal resolution; let $\psi: V \longrightarrow \mathbf{P}_1$ be a proper analytic map, whose generic fibre is a smooth elliptic curve. Let $C_u = \psi^{-1}(u)$, $\overline{\psi} = \psi \circ \sigma$, and $\overline{C}_u = \overline{\psi}^{-1}(u)$. Assume $\exists s \ni s: \mathbf{P}_1 \longrightarrow V$ is a section of $V \xrightarrow{\psi} \mathbf{P}_1$, and that V is smooth along $s(\mathbf{P}_1)$; let $\overline{s}$ be the corresponding section of $\overline{V} \xrightarrow{\overline{\psi}} \mathbf{P}_1$. Now assume $p_g(\overline{V}) = n \geqslant 1$, and $\overline{V}$ is a minimal surface; $\overline{s} \cdot \overline{C}_u = 1$, $\overline{s}^2 = -(n+1)$, $\overline{C}_u^2 = 0$. $V \xrightarrow{\psi} \mathbf{P}_1$ is called an *elliptic fibration* with a section (see **[K]**). We also require that $\sigma(D) =$ a point for all irreducible curves $D \subset \overline{V}$, with $\overline{s} \cdot D = 0 = \overline{C}_u \cdot D$.

Let $H^2(\overline{V}; \mathbf{Z})_0 = (\mathbf{Z}c_1(\overline{C}_u) + \mathbf{Z} \cdot c_1(\overline{s}))^\perp$, where $c_1(D)$ denotes the 1st Chern class of a divisor; this is an orthogonal direct summand of $H^2(\overline{V}; \mathbf{Z})$ and hence must be a unimodular lattice. It is uniquely characterized by the integer n; fix a copy and call it H; set $\Lambda = H \oplus (\mathbf{Z}c + \mathbf{Z}s)$, where $c \cdot H = s \cdot H = 0$, $c^2 = 0$, $s \cdot c = 1$, $s^2 = -(n+1)$.

Let $\varphi: H^2(\overline{V}; \mathbf{Z}) \longrightarrow \Lambda$ be an isomorphism of unimodular lattices $\varphi(c_1(\overline{C}_u)) = c$, $\varphi(c_1(\overline{s})) = s$, $\varphi(c_1(K_{\overline{V}})) = (n-1) \cdot c$ where $K_{\overline{V}}$ is the canonical divisor, and hence $\varphi(H^2(\overline{V}; \mathbf{Z})_0) = H$. Following **[P-S, Šaf]**, we will call φ a marking, and the pair (V, φ) a *marked surface* of type (H, Λ, c, s). Let $L_V \subset H^2(\overline{V}; \mathbf{Z})_0$ be the euclidean sublattice generated by all elements of the form $c_1(D)$, $D > 0$, $\sigma(D) =$ a point; and let $\widetilde{H}_V = (L_V)^\perp$; L_V is negative definite. For any euclidean lattice E, we let $O(E)$ be the orthogonal group of E.[1] This is a linear algebraic group defined over $\mathbf{Q}$; let $O(E)_\mathbf{Z} = \{g \in O(E)_\mathbf{R} \mid gE = E\}$. For any $\alpha \in E$, $\alpha^2 = -2$, let

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[1] The pairing on E may be indefinite.

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$S_\alpha \in O(E)_{\mathbf{Z}}$ be the reflection defined by $S_\alpha(\beta) = \beta + (\alpha \cdot \beta)\alpha$, $\forall \beta \in E$; and let $\mathfrak{R}(E) \subset O(E)_{\mathbf{Z}}$ be the subgroup generated by the reflections.

Let (V_i, φ_i), $i = 1, 2$, be two marked surfaces, f: $V_1 \longrightarrow V_2$ an isomorphism, $\overline{f}$ its unique lift to $\overline{V}_1 \longrightarrow \overline{V}_2$. Assume that $\exists g \in \mathfrak{R}(L)$, $\ni \varphi_1 \circ \overline{f}^* = \overline{g} \circ \varphi_2$, where $\overline{g}$ is the obvious extension of g to Λ; then we say that the marked surfaces are equivalent, $(V_1, \varphi_1) \sim (V_2, \varphi_2)$.

Let $\mathcal{J} = \mathcal{J}(H, \Lambda, c, s)$ *be the set of marked surfaces of type* (H, Λ, c, s), *modulo this equivalence.* Let $\Gamma = O(H)_{\mathbf{Z}}$, we define an action of Γ on $\mathcal{J}$ by: $\gamma \cdot (V, \varphi) = (V, \overline{\gamma} \circ \varphi)$, where $\gamma \in \Gamma$, and $\overline{\gamma}$ is its extension to Λ, defined by $\overline{\gamma}(c) = c$, $\overline{\gamma}(s) = s$.

It can be shown that $\mathcal{J}$ is a smooth analytic space, Γ acts properly discontinuously on $\mathcal{J}$, and $\mathcal{J}/\Gamma = M$ is irreducible.[2] Moreover, if $n \geqslant 2$ then for marked surfaces (V_i, φ_i), $i = 1, 2$: $\exists f$, $V_1 \xrightarrow{f} V_2$, $\ni f$ is a birational isomorphism with $f(s_1) = s_2$ iff $\exists \gamma \in \Gamma$, such that $\gamma \cdot (V_1, \varphi_1) \sim (V_2, \varphi_2)$.

2. Let $\mathrm{Gr}(n, H_{\mathbf{C}})$ be the Grassmannian of n-planes in $H_{\mathbf{C}}$; $\xi \in \mathrm{Gr}(n, H_{\mathbf{C}})$ corresponding to the n-plane $F(\xi) \subset H_{\mathbf{C}}$. Following Griffiths [G], let $\check{D}(H) = \{\xi \in \mathrm{Gr}(n, H_{\mathbf{C}}) | u \cdot v = 0, \forall u, v \in F(\xi)\}$; let $D(H) = \{\xi \in \check{D}(H) | \overline{u} \cdot u > 0, \forall u \in F(\xi) - \{0\}\}$. $D(H)$ is an open subset of $\check{D}(H)$, homogeneous under the action of $O(H)_{\mathbf{R}}$; Γ acts properly discontinuously on $D(H)$. Let $z = (V, \varphi) \in \mathcal{J}$ be a marked surface; and $H^{2,0}(V) \subset H^2(\overline{V}, \mathbf{C})$ be the subspace corresponding to holomorphic $(2, 0)$ forms. Let φ: $H^2(\overline{V}, \mathbf{C}) \longrightarrow \Lambda_{\mathbf{C}}$ also denote the extension of φ to $H^2(\overline{V}, \mathbf{C})$; clearly $\varphi(H^{2,0}(V)) \subset H_{\mathbf{C}}$, since $\varphi^{-1}(c)$, $\varphi^{-1}(s)$ both correspond to algebraic cycles. Define $\Phi(z) = \xi \in D(H)$ by $\varphi(H^{2,0}(V)) = F(\xi)$; clearly $\Phi(\gamma \cdot z) = \gamma \cdot \Phi(z)$.

By fundamental results of Griffiths, Φ is holomorphic. Let us assume $n \geqslant 2$.

MAIN THEOREM. *There is a* Γ*-invariant dense open subset of* $\mathcal{J}$ *on which* Φ *is injective.*

REMARK. This is equivalent to a "degree one" statement.

Following the philosophy in [P-S, Šaf], the essential idea is to reduce the problem to a "special" sublocus of $\mathcal{J}$.

3. Let $(V, \varphi) \in \mathcal{J}$, assume $n \geqslant 2$, then the elliptic fibration ψ: $V \longrightarrow \mathbf{P}_1$ is uniquely determined. Let $C_{a_1}, \ldots, C_{a_r}$ be the singular fibres of ψ, so that C_u is smooth $\forall u \in \mathbf{P}_1' = \mathbf{P}_1 - \{a_1, \ldots, a_r\}$; choose $a \in \mathbf{P}_1'$, and $e_1, e_2 \in H_1(C_a, \mathbf{Z})$, $e_1 \cdot e_2 = 1$. Let τ_i be a local parameter at a_i, the loop γ: $[0, 1] \longrightarrow \mathbf{P}_1'$ defined by $\tau_i(\gamma(t)) = \epsilon \exp\{2\pi i t\}$ defines a conjugacy class in $\pi_1(\mathbf{P}_1', a)$, any member of

[2] It follows from the work of Kas that M is irreducible.

which will be referred to as a positive loop about a_i. Let T: $\pi_1(\mathbf{P}_1', a) \longrightarrow$ SL(2, **Z**) be the monodromy of $V \longrightarrow \mathbf{P}_1$, corresponding to the choice of a, and e_1, e_2. For $w \in$ SL(2, **Z**), we let $[w]$ denote its conjugacy class; C_{a_i} *is said to be of type* $[w]$ *if*: $T(\gamma) = [w]$ *for any positive loop* γ *about* a_i (see **[M]**).

We will call (V, φ) a *special elliptic surface* if $T(\pi_1(\mathbf{P}_1', a)) \subset \{\pm 1_2\}$, $1_2 = \binom{1\,0}{0\,1}$. Let $J_1 = \{z \in J | z = (V, \varphi)$ is a special elliptic surface$\}$. One can show that J_1 is a smooth analytic subvariety of J, invariant under Γ, and J_1/Γ is irreducible. These surfaces will play a role similar to the one played by special Kummer surfaces in **[P-S, Šaf]**; with one major exception: they will not be a dense subset.

4. Let Φ_1: $J/\Gamma \longrightarrow D/\Gamma$ be the map induced by $J \xrightarrow{\Phi} D$, $D = D(H)$, π: $D \longrightarrow D/\Gamma$ the qt. map. Assume there is a compact projective variety Y, divisors, E', $E'' \subset Y$, a proper surjective map g: $Y - E' - E'' \longrightarrow M = J/\Gamma$; the map $f = \Phi_1 \circ g$ extends to a proper holomorphic map f_1, f_1: $Y - E'' \longrightarrow D/\Gamma$. Moreover, assume $f^{-1}(J_1/\Gamma)$ is quasi-projective. This will all hold in our situation, due to the construction of M as a quotient of an open subset of a suitable Hilbert scheme; together with the results of **[G]**. Let $Z \subset J_1$ be an irreducible component, then we may write

$$\overline{\Phi(J)} \supset \Phi(J) \supset \overline{\Phi(J)} - N, \qquad \overline{\Phi(Z)} \supset \Phi(Z) \supset \overline{\Phi(Z)} - N_1$$

where $\overline{\Phi(J)}$, $\overline{\Phi(Z)}$, N, N_1 are all closed analytic subvarieties of D (see **[G]**).

Consider the following statements:

(a) $(d\Phi)_z$ is injective at each $z \in Z$,

(b) $\Phi^{-1}(\Phi(Z)) \subseteq \bigcup_{\gamma \in \Gamma} \gamma \cdot Z$,

(c) Φ/Z: $Z \longrightarrow \Phi(Z)$ is injective,

(d) $\{\gamma \in \Gamma | \gamma(Z) = Z\} = \{\gamma \in \Gamma | \gamma\overline{\Phi(Z)} \cap \overline{\Phi(Z)} = \overline{\Phi(Z)}\}$,

(e) $\Phi(Z) \nsubseteq \pi^{-1}(f_1(E' - E''))$.

The following is not difficult to prove:

LEMMA. (a)–(e) *imply the main theorem.*

Let us make some remarks concerning the proofs of (a)–(d).

(a) This may be proven by using the techniques of **[Kiĭ]**.

(b) This is proven by the same methods used by **[P-S, Šaf]** to characterize special Kummer surfaces, by means of their lattice of algebraic cycles.

(c) and (d) are reduced to the Torelli theorem for hyperelliptic curves, by means of the results contained in **[B]**. The idea of using the results in **[B]** may also be found in **[P-S, Šaf]**, however we use them somewhat differently. In order to understand our method of dealing with statement (e), we need to introduce some definitions.

By a *degeneration* of *elliptic fibrations* with section we will mean the following:

(i) connected analytic spaces $\mathcal{V}$, $\mathcal{D}$, $\dim_{\mathbf{C}}\mathcal{V} = 3$, $\dim_{\mathbf{C}}\mathcal{D} = 2$, proper maps, Ψ, ρ, p, making the following diagram commute.

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{\ \Psi\ } & \mathcal{D} \\ & {\scriptstyle\rho}\searrow \quad \swarrow{\scriptstyle p} & \\ & D_\epsilon = \{\tau \in \mathbf{C}^1 \mid |\tau| < \epsilon\} & \end{array}$$

Set $V_t = \rho^{-1}(t)$, $\Delta_t = p^{-1}(t)$, Δ_t is assumed to be connected, $\forall t \neq 0$; the generic fibre of Ψ is a smooth elliptic curve.

(ii) V_t, $\forall t \neq 0$, has at most rational double points; $\mathcal{D} - \Delta_0$ is smooth.

(iii) There is a holomorphic map $s\colon \mathcal{D} \longrightarrow \mathcal{V}$ with $\Psi \circ s = \mathrm{id}_{\mathcal{D}}$, i.e. a section. We will restrict our attention to the case where $\Delta_t \cong \mathbf{P}_1$, $\forall t \neq 0$. Let P denote the double covering of the Hirzebruch surface $\Sigma_2 \xrightarrow{p_2} \mathbf{P}_1$ branched along $B \in |3S_\infty + S|$, where S_∞ is the unique section of p_2 with $S_\infty^2 = -2$, and $S \sim S_\infty + 2F$, $F \sim p_2^{-1}(u)$; we require S_∞ to be an isolated component of B, thereby occurring with multiplicity 3. Let

$$\Psi_P\colon P \underset{\curvearrowleft\, s_P}{\longrightarrow} \mathbf{P}_1,$$

be the fibration induced by p_2, s_P the section defined by $S \subset B$. Each fibre of Ψ_P is a rational curve with a cusp, whose singularity has the form $z^2 = y^3$. P, together with its fibration, will be called a *connecting component*. The degeneration of elliptic fibrations with section

$$\begin{array}{ccc} \mathcal{V} & \underset{\curvearrowleft}{\xrightarrow{\ \Psi\ }} & \mathcal{D} \\ & {\scriptstyle\rho}\searrow \quad \swarrow{\scriptstyle p} & \\ & D_\epsilon & \end{array}$$

for which $\mathcal{V}/D_\epsilon - \{0\}$ has trivial monodromy as a fibration of surfaces with rational double points, will be called *stable* if the following holds:

(1) Δ_0 has only ordinary nodes, and p vanishes to multiplicity one on each component of Δ_0. Set $\Delta_0 = \bigcup \Gamma_i$, each $\Gamma_i \cong \mathbf{P}_1$, and $X_i = \Psi^{-1}(\Gamma_i)$. Each X_i occurs with multiplicity one.

(2) Each fibre of Ψ, when reduced, is an irreducible curve. Set $C_u = \Psi^{-1}(u)$, $\forall u \in \mathcal{D}$; $\mathcal{V}$ is smooth along $s(\mathcal{D})$.

(3) If the generic fibres of $X_i \longrightarrow \Gamma_i$ and $X_j \longrightarrow \Gamma_j$ are smooth then $\Gamma_i \cap \Gamma_j = \varnothing$.

(4) If the generic fibre of $X_i \longrightarrow \Gamma_i$ is smooth, $\overline{\Delta_0 - \Gamma_i} \cap \Gamma_i = \{q_1, \ldots, q_l\}$, we let μ_i be the number of singular fibres of $X_i|_{\Gamma_i - \{q_1,\ldots,q_l\}}$, *properly* counted. If the generic fibre of $X_i \longrightarrow \Gamma_i$ is singular, set $\mu_i = 0$. We require that $\Sigma_j \mu_j = 12(n+1)$, where $p_g(V_{t_0}) = n$, $t_0 \neq 0$.[3]

(5) If the generic fibre of $X_i \longrightarrow \Gamma_i$ is smooth, and $p_g(X_i) = 0$ then Γ_i meets at least two other components of Δ_0; if Γ_i meets exactly two other components, then $\mu_i > 0$.

(6) If the generic fibre of $X_k \longrightarrow \Gamma_k$ is singular, then X_k is a connecting component and Γ_k meets exactly two other components of Δ_0, $\Gamma_{k'}$, $\Gamma_{k''}$. Set $\{q'\} = \Gamma_{k'} \cap \Gamma_k$, $\{q''\} = \Gamma_{k''} \cap \Gamma_k$, then the generic fibres of $X_{k'} \longrightarrow \Gamma_{k'}$, $X_{k''} \longrightarrow \Gamma_{k''}$ are smooth; and let $C_{q'}$ be of type $[w_1]$, $C_{q''}$ of type $[w_2]$. Let $w_i = s_i u_i$, $i = 1, 2$ be their Jordan decomposition into semisimple and unipotent elements, s_i semisimple, u_i unipotent. Then, $u_1 \neq 1$ iff $u_2 \neq 1$; and $[s_1^{-1}] = [s_2] \neq [1]$.

THEOREM (STABLE REDUCTION). *Let $\mathcal{V}$, $\mathcal{D}$ etc. be a degeneration of elliptic fibrations with section, such that $\mathcal{V}|_{D'_\epsilon}$, $D'_\epsilon = D_\epsilon - \{0\}$ has trivial monodromy as a fibration of surfaces with rational double points, then after a base extension $\varphi\colon D_{\epsilon_1} \longrightarrow D_\epsilon$, $\mathcal{V} \times_\varphi D_{\epsilon_1}$ is D_{ϵ_1}-birationally equivalent to a stable degneration.*

This theorem is the key element in obtaining a proof of statement (e), above. To use it, we let the Y, used to dominate $\mathcal{M}$, be obtained from a certain Hilbert scheme, $\mathcal{X} \longrightarrow Y$ the universal surface over Y, the fibres of $\mathcal{X}|Y - E' - E''$ being elliptic surfaces with at most rational double points $E' - E'' = E_1 \cup \cdots \cup E_m \cup \widetilde{E}$, where $\mathcal{X}|_{E_i}$ has only fibres which fulfill properties (1)–(6) above, and $f_1(\widetilde{E}) \subset \bigcup_j f_1(E_j)$. Due to properties (1)–(6), if we let E be any one of the E_i, $\exists$ euclidean sublattices $H_j \subset H$, $1 \leqslant j \leqslant r$ each H_j having a nonzero even number of positive eigenvalues, $(H_1 + \cdots + H_r)^\perp$ being negative definite, $H_j \subset H_i^\perp$ $\forall i \neq j$; and a holomorphic map of $\widetilde{D}/\widetilde{\Gamma} \xrightarrow{\widetilde{F}} D/\Gamma$, where $\widetilde{D} = \Pi_i D(H_i)$, $\widetilde{\Gamma} = \Pi O(H_i)_{\mathbf{Z}}$, $\widetilde{F}$ induced by the inclusion $(H_1 + \cdots + H_r) \subset H$. Moreover, $\widetilde{F}(\widetilde{D}/\Gamma) \supset f_1(E)$.

Let $\Gamma_Z = \{\gamma \in \Gamma | \gamma(Z) = Z\}$; $\mathcal{A}_{\mathbf{R}}(\Gamma_Z)$ its Zariski closure in $O(H)_{\mathbf{R}}$; then $\Phi(Z) \subset \pi^{-1}(f_1(E))$ implies that some conjugate of $\mathcal{A}_{\mathbf{R}}(\Gamma_Z)$ must leave an orthogonal decomposition $H_{\mathbf{R}} = H' + H''$, invariant, each H', H'' having a nonzero number of positive eigenvalues. It may be verified that this is impossible.

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[3] It follows from this, and condition (6), that $\Sigma_i p_g(X_i) = n$.

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