rapidly than exponentially, thereby accounting for the uniqueness of solutions.

In short, Professor Arnold is one of the truly great stars of mathematics, and in this outstanding text, he shares his knowledge and understanding with us.

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BULLETIN (New Series) OF THE
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Methods of modern mathematical physics, vol. III, *Scattering theory*, by Michael Reed and Barry Simon, Academic Press, New York, 1979, xv + 463 pp., $42.00.

Methods of modern mathematical physics, vol. IV, *Analysis of operators*, by

Michael Reed and Barry Simon, Academic Press, New York, 1978, xv + 396 pp., $34.00.

1. With a knowledge of the equations of physics and with sufficient mathematical insight, one should be able to derive the physical properties of the world we see around us. The program would be to classify the elementary particles and analyze the various forces between them. Their motions should then follow from the laws of quantum mechanics.

The difficulty, of course, is that the classification is not complete and the forces are not known. There is a traditional division of forces into strong, electromagnetic, weak, and gravitational interactions. The first three of these play a role on the submicroscopic level and are presumably described by quantum fields. The present is a period of intense speculation about the nature of the fields responsible for the strong and weak forces. There is already a detailed theory of the electromagnetic field, but even that is not on a completely rigorous footing. So a complete mathematical description of nature is for the future.

It is possible, however, to come rather close to this ideal even now. Consider a world consisting of electrons and nuclei. Ignore most of the quantum mechanical features of the electromagnetic field; in fact, direct attention to the part of the electric field given by Coulomb's inverse square law. Treat this system with quantum mechanics, in the nonrelativistic approximation in which there is no particle creation. Add a few refinements, such as spin and the exclusion principle. The result should be a fairly good description of our world. This model should describe most physical and chemical properties of materials. It should explain why tables are solid and fire is hot, why the sky is blue and grass is green. It should not explain nuclear energy, radioactivity, or why apples fall, since these involve the other forces.

The task remains of carrying out this derivation. Is there anything interesting to say before going to the computer? There had better be, if we want to be able to make much sense out of physics. The series *Methods of modern mathematical physics* is an ambitious attempt to survey recent progress toward a rigorous qualitative description of quantum mechanical motion. (The series also contains considerable material on related mathematical problems, but the central theme is quantum mechanics.) The first two volumes [5] [6] in the series dealt with some preliminary functional analysis and with the determination of the time evolution. Volumes III and IV under review are companion volumes dealing with scattering and spectral theory. They describe the different possible kinds of motions of the individual particles or atoms or molecules.

Of course, in order to make full contact with the world of experience, one also needs models of bulk matter. Thus far in the series the closest approach to a description of these models is a discussion of independent electrons in a crystalline solid in volume IV. (The discrete translational symmetry is responsible for the energy gaps that lie behind the distinction between insulator and metal.) There is nothing so far on probabilistic models involving the temperature concept (statistical mechanics), but that would come logically in later volumes.

2. It is difficult or impossible to give a fully intuitive account of quantum mechanics. The theory has features which are paradoxical or subjectivist, depending on how you choose to look at it. Nevertheless, its equations have stood repeated experimental test. It is possible to doubt quantum mechanics, but one then needs to replace it with some other theory that gives similar equations of motion. This can be done **[4]**, but most physicists, having once passed through the crisis of a first encounter with quantum mechanics, are content with the orthodox account. Certainly there is no alternative in sight that would produce simpler calculations.

In quantum mechanics states of a physical system are given by one-dimensional subspaces of a Hilbert space $\mathcal{H}$. Thus a nonzero vector ψ in $\mathcal{H}$ determines a state, and two such vectors determine the same state precisely when they are multiples of each other. Physical quantities correspond to selfadjoint linear operators A acting in the Hilbert space. In a state given by a unit vector ψ each of these quantities becomes a random variable, with expectation given by $\langle \psi, A\psi \rangle$, where the bracket denotes the inner product. If the operator happens to be of the form W^*W, where W^* is the adjoint of W, then this expectation is actually the square of a Hilbert space norm

$$\langle \psi, W^*W\psi \rangle = \langle W\psi, W\psi \rangle = \| W\psi \|^2 \tag{1}$$

and so it is positive. Many estimates in the theory involve expectations of this form. One special case occurs when the operator W is of the form $W\psi = \phi\langle \chi, \psi \rangle$ for fixed vectors ϕ and χ. Then the quantity to be estimated is $\| W\psi \|^2 = \|\phi\|^2 |\langle \chi, \psi \rangle|^2$. A compact operator is one that can be approximated in norm by sums of operators of this form, and estimates on $|\langle \chi, \psi \rangle|^2$ often transfer to estimates on $\| W\psi \|^2$ for general compact operators W.

Quantum mechanical motion is determined by the selfadjoint operator H that corresponds to the total energy of the system. (It is customary to use the letter H in this context. Here it stands for Hamilton instead of Hilbert. There is yet another h in the theory, due to Planck, but I have suppressed that.) The unitary operator $\exp(-itH)$ acts on vectors (hence on states) and describes their evolution in a time interval of length t. Thus if ψ gives the state at time zero, then $\exp(-itH)\psi$ gives the state at time t. The expectation of W^*W in this state is then $\| W \exp(-itH)\psi \|^2$.

Let $E(\lambda)$ be the projection onto the subspace where $H \leqslant \lambda$ and $P(\lambda)$ be the projection onto the subspace where $H = \lambda$. (Thus $P(\lambda) \neq 0$ if and only if λ is an eigenvalue of H, and then $P(\lambda)$ is the projection onto the corresponding eigenspace.) For all vectors χ and ψ in $\mathcal{H}$,

$$\langle \chi, \exp(-itH)\psi \rangle = \int_{-\infty}^{\infty} \exp(-it\lambda)\langle \chi, dE(\lambda)\psi \rangle. \tag{2}$$

It follows that the time average of $|\langle \chi, \exp(-itH)\psi \rangle|^2$ is

$$\frac{1}{2T}\int_{-T}^{T} |\langle \chi, \exp(-itH)\psi \rangle|^2 \, dt = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\sin(T(\lambda_1 - \lambda_2))}{T(\lambda_1 - \lambda_2)} \langle \psi, dE(\lambda_1)\chi \rangle \langle \chi, dE(\lambda_2)\psi \rangle. \tag{3}$$

Let $T \to \infty$ and apply the dominated convergence theorem. The only contribution to the limit is from the diagonal $\lambda_1 = \lambda_2$. The calculation gives

$$\frac{1}{2T}\int_{-T}^{T} |\langle \chi, \exp(-itH)\psi\rangle|^2\, dt \to \sum_{\lambda} |\langle \chi, P(\lambda)\psi\rangle|^2 \tag{4}$$

as $T \to \infty$, where the sum is over the eigenvalues of H. The generalization of this result to compact operators W is

$$\frac{1}{2T}\int_{-T}^{T} \| W\exp(-itH)\psi\|^2\, dt \to \sum_{\lambda} \| WP(\lambda)\psi\|^2 \tag{5}$$

as $T \to \infty$.

This identity lies behind the correspondence between different kinds of spectrum and different kinds of motion. The eigenvalues of H are called the point spectrum and the space spanned by the eigenvectors is called the point subspace. A state given by a vector ψ in the point subspace is called a bound state, in analogy to a bound orbit in classical mechanics. Such a vector ψ has the decomposition $\psi = \Sigma_\lambda P(\lambda)\psi$. Whenever $P(\lambda)\psi \neq 0$, it determines a state which is time invariant. The identity (5) says that the time average of the expectations in the states $\exp(-itH)\psi$ approaches a sum of expectations in time invariant states $P(\lambda)\psi$.

Since the Hilbert space of a quantum mechanical particle is infinite dimensional (countably infinite), there is also the possibility that H has continuous spectrum. The continuous subspace is defined as the orthogonal complement of the eigenvectors, and the continuous spectrum is the spectrum of H restricted to this subspace. If the state is given by a vector ψ in the continuous subspace, then the time average in (5) approaches zero. This corresponds to a situation where the particle wanders away from its starting point and returns infrequently, if at all.

There is a further distinction between absolutely continuous and singular continuous spectrum. A vector ψ is in the absolutely continuous (or singular continuous) subspace if the measure $\langle \psi, dE(\lambda)\psi\rangle$ is absolutely continuous (or singular continuous) with respect to Lebesgue measure. If ψ is in the absolutely continuous subspace, then we may write its measure in terms of a density as $\langle \psi, dE(\lambda)\psi\rangle = \langle \psi, \delta(H - \lambda)\psi\rangle d\lambda$. The expression $\delta(H - \lambda)$ does not define an operator with values in the Hilbert space, but makes sense as a positive quadratic form with values denoted by $\langle \psi, \delta(H - \lambda)\psi\rangle$.

If χ and ψ are vectors in the absolutely continuous subspace, then the matrix element in (2) approaches zero as $t \to \infty$, by the Riemann-Lebesgue lemma. Again this fact may be generalized to compact operators W, giving

$$\| W\exp(-itH)\psi\|^2 \to 0 \tag{6}$$

as $t \to \infty$ for ψ in the absolutely continuous subspace. In practice the expectation in (6) is a measure of how likely the particle is to be in a fixed region. The fact that this expectation becomes and remains small show that ψ has the behavior of a scattering state, in which the particle eventually leaves every fixed region and never returns. (With singular continuous spectrum, on the other hand, the particle may wander back from time to time. One hopes this complicated misbehavior is rare in physics.)

The task is thus to find out which kind of behavior occurs in a specific physical situation. The most useful approach is perturbation theory, and indeed the main mathematical topic of Volumes III and IV is perturbation of spectra. We have an operator H_0 with known spectral properties. It describes motion without interaction, or with only part of the interaction. The operator of interest is $H = H_0 + V$, where V is the interaction. We have some kind of estimate on V and wish to find the spectral properties of H. For instance, if H_0 has eigenvalues, we want to find corresponding eigenvalues of H. If H_0 has absolutely continuous spectrum, we want to find at least part of the space on which H has absolutely continuous spectrum. (It turns out that the singular continuous spectrum is highly unstable, and there is no good theory in the third case. This is another reason for wishing that it would go away.)

3. Volume III is organized about the long time approach to the problem, that is, scattering theory. The operator H_0 has absolutely continuous spectrum. One wants to show that for every vector ϕ in $\mathcal{H}$ there is a vector ψ in $\mathcal{H}$, such that

$$\|\exp(-itH)\psi - \exp(-itH_0)\phi\| \to 0 \tag{7}$$

as $t \to \infty$. This would show that the states $\exp(-itH)\psi$ have a simple time evolution, at least in the distant future. One also wants to prove completeness of the ψ, in the sense that they span the absolutely continuous subspace for H, and to prove that the singular continuous spectrum is empty. Then one would know that the only states are bound states and scattering states.

The norm convergence in (7) is equivalent to requiring that

$$\begin{aligned} &\langle \exp(-itH)\psi - \exp(-itH_0)\phi, \exp(-itH_0)\chi\rangle \\ &\qquad = i\int_0^t \langle \exp(-itH)\psi, V\exp(-itH_0)\chi\rangle dt - \langle \phi - \psi, \chi\rangle \end{aligned} \tag{8}$$

approaches zero as $t \to \infty$, uniformly on sets of χ with bounded norm. (The last line in (8) comes from the fundamental theorem of calculus; the factor $V = H - H_0$ arises from differentiating the inner product of exponentials. I am using the physicist's convention that inner products are linear on the right.) To prove completeness, we can start with ψ and try to recover ϕ from the formula

$$i\int_0^\infty \langle \exp(-itH)\psi, V\exp(-itH_0)\chi\rangle dt = \langle \phi - \psi, \chi\rangle. \tag{9}$$

One approach to proving convergence of the integral is to decompose $V = W_1^* W_2$ into two factors of roughly the same size and estimate

$$\begin{aligned} &\int_0^\infty |\langle \exp(-itH)\psi, V\exp(-itH_0)\chi\rangle| dt \\ &\qquad \leqslant \int_0^\infty \|W_1 \exp(-itH)\psi\| \, \|W_2 \exp(-itH_0)\chi\| dt \\ &\qquad \leqslant \left(\int_0^\infty \|W_1 \exp(-itH)\psi\|^2 \, dt\right)^{1/2} \left(\int_0^\infty \|W_2 \exp(-itH_0)\chi\|^2 \, dt\right)^{1/2}. \end{aligned} \tag{10}$$

This inequality (10) shows that the scattering problem is closely related to time decay estimates on expectations of the form $\|W \exp(-itH)\psi\|^2$. One needs ways of getting such estimates.

In Volume III the Kato-Birman method gets the most extensive treatment. This is a particularly nice method, because the estimates are in terms of Hilbert space invariants such as traces. The starting point is the observation that if ψ is a vector in the absolutely continuous subspace with the additional property that $\langle\psi, \delta(H-\lambda)\psi\rangle$ as a function of λ is bounded by a constant M, then

$$\int_{-\infty}^{\infty} |\langle\chi, \exp(-itH)\psi\rangle|^2 \, dt = 2\pi \int_{-\infty}^{\infty} |\langle\chi, \delta(H-\lambda)\psi\rangle|^2 \, d\lambda$$
$$\leqslant 2\pi \int_{-\infty}^{\infty} \langle\chi, \delta(H-\lambda)\chi\rangle\langle\psi, \delta(H-\lambda)\psi\rangle \, d\lambda \leqslant 2\pi M \|\chi\|^2. \tag{11}$$

This has a generalization to operators W with tr $W^* W < \infty$:

$$\int_{-\infty}^{\infty} \|W \exp(-itH)\psi\|^2 \, dt = 2\pi \int_{-\infty}^{\infty} \|W\delta(H-\lambda)\psi\|^2 \, d\lambda$$
$$\leqslant 2\pi M \operatorname{tr} W^* W. \tag{12}$$

Such an inequality is just what is needed for scattering theory. However, the Kato-Birman method has the weakness that ψ must be assumed at the outset to be in the absolutely continuous subspace.

Volume III applies this method and others to scattering in a large variety of situations. It begins with classical mechanics, two-body quantum mechanics, and N-body quantum mechanics. There is a detailed treatment of central forces. One irritation in the subject is that the unshielded Coulomb force doesn't fall under the general theory. It has such a slow decrease at infinity that the particle never fully escapes its influence. There is an extensive discussion of the modifications of the theory that are necessary to take care of such a long range force. Once the authors had covered quantum mechanics it was surely tempting to look for other areas of physics where the same ideas apply. Thus optical and acoustical scattering are treated by trace estimates and by the Lax-Phillips theory, based on finite propagation speed. There are also sections on the linear Boltzmann equation, nonlinear wave equations, spin waves, and even on quantum fields.

The phase space approach to two-body quantum scattering due to Enss appeared just in time to be included in a final section of Volume III. The starting point is to assume that the particle is not bound. If one waits long enough, the particle will eventually wander far from the scattering center (as a consequence of the fact that the time average in (5) approaches zero). Then a phase space decomposition can be used to prove that the particle has good scattering behavior.

Volume IV is organized around the division into different kinds of spectra. There is more on the absolutely continuous spectrum, and most of this material is also relevant to scattering theory. One major topic is the smooth operator theory of Kato. Its basic abstract estimate is a bound on the time integral or the energy integral in (12) by a multiple of $\|\psi\|^2$. Thus there is no special assumption on ψ and the burden falls on the operator W.

Such an estimate may be obtained if one has sufficient information on spectral properties of H. For instance, assume that vectors $W^*\phi$ in the range of W^* are in the absolutely continuous subspace for H and satisfy $\langle W^*\phi, \delta(H - \lambda)W^*\phi\rangle \leqslant C\|\phi\|^2$ for some constant C. This is equivalent to a bound $\|W\delta(H - \lambda)W^*\| \leqslant C$ on the operator norm of $W\delta(H - \lambda)W^*$. If this holds, then the energy integral satisfies

$$\begin{aligned}\int_{-\infty}^{\infty} \|W\delta(H - \lambda)\psi\|^2\, d\lambda \\ &\leqslant \int_{-\infty}^{\infty} \|W(\delta(H - \lambda))^{1/2}\|^2\|(\delta(H - \lambda))^{1/2}\psi\|^2\, d\lambda \\ &= \int_{-\infty}^{\infty} \|W\delta(H - \lambda)W^*\|\langle\psi, \delta(H - \lambda)\psi\rangle\, d\lambda \leqslant C\|\psi\|^2. \quad (13)\end{aligned}$$

Once we have this bound on the energy integral, the same bound on the time integral follows automatically. (A reader who is suspicious of my use of δ functions may consult Volume IV. The reasoning is made precise there by obtaining estimates that are uniform for a family of approximate δ functions.)

How does one get such estimates in quantum mechanics? One method is based on ideas of Putnam, Kato and Lavine. In this approach one looks for an increasing operator A such that

$$\begin{aligned}\frac{d}{dt}\langle\exp(-itH)\psi, A\exp(-itH)\psi\rangle \\ &= \langle\exp(-itH)\psi, i(HA - AH)\exp(-itH)\psi\rangle \\ &\geqslant \|W\exp(-itH)\psi\|^2 \geqslant 0. \quad (14)\end{aligned}$$

If A is bounded, then the time integral in (12) satisfies

$$\int_{-\infty}^{\infty} \|W\exp(-itH)\psi\|^2\, dt \leqslant 2\|A\|\ \|\psi\|^2. \quad (15)$$

Lavine extended this reasoning to certain unbounded operators and applied it to a variety of situations. In his work the problem often reduces to verifying an algebraic condition: the commutator $i(HA - AH)$ occurring in the time derivative must be positive.

As we have seen, the authors present a number of different approaches to scattering theory. Volume III contains the Kato-Birman trace theory and the Enss phase space analysis, and Volume IV has the Kato-Putnam-Lavine approach and still another set of estimates due to Agmon. How are we to compare these approaches? The Kato-Birman theory is elegant but does not provide complete spectral information. The method of Enss is perhaps the simplest way to prove the absence of singular continuous spectrum. However the more explicit decay estimates in Volume IV are interesting in their own right, since they also tell us how fast the particle is escaping the scattering center. A particularly nice feature of both the Enss and Lavine methods is that the estimates have a direct intuitive content.

4. Volume IV contains considerable material on perturbation of the point spectrum. There is an account of the regular case when the perturbation series for an eigenvalue converges to the correct answer. But the authors also want to push perturbation theory to its limits. Thus when the series diverges, they look to see if a summability method will give the correct answer. Or when the perturbed eigenvalue does not even exist, they look for a resonance. A resonance is interpreted here as a pole of an analytic continuation of certain matrix elements $\langle\psi, (H - z)^{-1}\psi\rangle$. The proper mathematical setting for resonances has always been somewhat mysterious, but it would seem that this definition is not as closely linked to scattering theory as one would like. Still, it provides a framework in which the authors can present a nice application of dilation analyticity techiques.

When the eigenvalues arise from a perturbation of the continuous spectrum, one is beyond the normal scope of perturbation theory. There is still much to say in the context of quantum mechanics. For example, there is the beautiful estimate of the number of bound state eigenvalues, due to Cwickel, Lieb, and Rosenbljum. The authors follow Lieb in giving a proof of this result that is based on Wiener measure.

Let us look at this example in more detail, since it illustrates some typical features of mathematical physics. The setting is the Hilbert space $\mathcal{H} = L^2(\mathbf{R}^3)$ and selfadjoint operators $H_0 = -\Delta$ and $V =$ multiplication by v. Here v is a real function in $L^{3/2}(\mathbf{R}^3)$. Assume for simplicity that $v \leqslant 0$. The theorem states that the number $n(v)$ of strictly negative eigenvalues of $H = -\Delta + V$ is bounded by a multiple of the $L^{3/2}$ norm,

$$n(v) \leqslant c\int(-v(x))^{3/2}d^3x. \tag{16}$$

One's first reaction may be that this is just another obscure fact about eigenvalues of partial differential operators. But let us compute the volume in classical phase space of the set $N(v)$ where $p^2 + v(x) < 0$. For each fixed x the section of this set is the ball $|p| < (-v(x))^{1/2}$, so the volume is

$$\int\int_{N(v)} d^3p\,d^3x = (4/3)\pi\int(-v(x))^{3/2}d^3x. \tag{17}$$

Thus the inequality is actually a beautiful illustration of the heuristic principle that the number of quantum mechanical bound states is proportional to the volume of the bound states in classical phase space.

If this were the end of the story, it would be a striking vindication of the heuristic principle. However, the argument above does not depend in any way on the dimension of space. The authors point out that the corresponding inequality $n(v) \leqslant c_n\int(-v(x))^{n/2}d^nx$ for n dimensional space is definitely false for $n = 1$ and $n = 2$. Thus it requires something more to see what is going on. In particular, any serious analysis of the effect must necessarily involve the fact that we live in three dimensional space.

Volume IV continues with the dilation analyticity approach of Balslev and Combes, a seemingly miraculous method of uncovering hidden information about the spectrum when the forces have some homogeneity property. It concludes with a number of other topics, including estimation of eigenfunctions, nondegeneracy of the ground state, absence of positive eigenvalues, and asymptotic distribution of eigenvalues. The two volumes together contain forty sections, each of which is in effect a chapter on a different subject. Every section has extensive notes with references to the original papers. There are 326 problems. A number of these outline the contents of research papers.

5. The method that is used throughout these volumes is to present abstract operator theory combined with substantial concrete application. (In this review I have been able to present only an outline of the operator theory, and almost nothing of the estimates that are the basis of the applications.) There is in addition an immense amount of physics lore along the way, and many nice mathematical counterexamples. It sometimes takes an effort to see the physics behind the estimates. For instance, certain scattering theory estimates break down near zero energy, and a reader could forget that this has a simple interpretation in terms of the fact that a slowly moving particle is going to have a comparatively hard time escaping from the region where the forces are strong. Other estimates fail in the classical limit. These have a deeper interpretation as showing that particles escape a region of strong attractive forces by barrier penetration, a uniquely quantum mechanical effect.

The spectral classification and perturbation theory unify the abstract part of the theory. Is there a unifying idea for the estimates? One might point out the role of the dilation operator, which involves both of the phase space quantities, position and momentum. It seems to be useful in distinguishing between incoming and outgoing particles. The increasing operator A in Lavine's theory is closely related to the dilation operator. Similar ideas occur in the phase space analysis of Enss. The dilation operator is an essential part of the analyticity method of Balslev and Combes. It also helps O'Connor, Combes, and Thomas estimate eigenfunctions. Perhaps this points to a common algebraic structure behind many of the estimates. (There is now new evidence for this in a recent article of Mourre [**3**] on the role of the dilation operator in phase space analysis.)

The value of the Reed and Simon volumes is that they provide a bridge between abstract functional analysis and concrete problems of physics. There are no other books with the same scope. The classic work of Kato [**2**] is organized along more strictly mathematical lines. Other recent works on rigorous quantum scattering theory, such as the text by Amrein, Jauch, and Sinha [**1**], have more limited objectives. The Reed and Simon series is the one place where the recent discoveries of mathematical physics are consolidated. Its volumes are for anyone who has encountered the frustration and fascination of quantum mechanics and wants to begin the serious task of learning the mathematics behind it. This will never be an easy task, but *Methods of modern mathematical physics* will make it less forbidding.

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