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COHOMOLOGY OF SUBGROUPS OF FINITE INDEX OF $SL(3, \mathbf{Z})$ AND $SL(4, \mathbf{Z})$

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Let $SL(n, \mathbf{Z})(p)$ for $n \geqslant 2$ and $p \geqslant 3$ denote the kernel of the reduction modulo p: $SL(n, \mathbf{Z}) \longrightarrow SL(n, \mathbf{Z}/p)$. The integral homology and cohomology of $SL(3, \mathbf{Z})(3)$ have been entirely computed in [1]. On p. 28 the authors make a conjecture that would imply that $H^3(SL(3, \mathbf{Z})(p), \mathbf{Z}) \simeq H_1(T/SL(3, \mathbf{Z})(p), \mathbf{Z})$, where T is the Tits building associated to $SL(3, \mathbf{Q})$, $SL(3, \mathbf{Z})(p)$ acts naturally on it, and p is prime. This conjecture is wrong.

THEOREM 1. *There is a natural surjective map*

$$H^3(SL(3, \mathbf{Z})(p), \mathbf{R}) \longrightarrow H_1(T/SL(3, \mathbf{Z})(p), \mathbf{R}) \oplus [H_1(X(p), \mathbf{R})]^k.$$

Here $p \geqslant 3$. $X(p)$ is the closed Riemann surface obtained by adding in the cusps to the quotient of the upper half-plane by $SL(2, \mathbf{Z})(p)$, and k is the number of orbits of maximal parabolic subgroups of $SL(3, \mathbf{Q})$ under conjugation by $SL(3, \mathbf{Z})(p)$. If p is prime, $k = p^3 - 1$.

Let $h^i(A) = \dim H^i(A, \mathbf{R})$. Since the euler characteristic of $SL(3, \mathbf{Z})$ is 0 (for example, see [2]) and $H^1(SL(3, \mathbf{Z})(p), \mathbf{R}) = 0$ by [3], Theorem 1 also gives a lower bound on $h^2(SL(3, \mathbf{Z})(p))$.

My original proof of Theorem 1 was along the lines described below for Theorem 2. With the help of A. Borel, we could prove the natural generalization of Theorem 1 for arithmetic subgroups of any $\mathbf{Q}$-rank 2 group G. The proof involves the manifold with corners M for G, the Leray spectral sequence for $\partial M \longrightarrow$ Tits building (G), and the vanishing of h^1.

The kernel of the map in Theorem 1 probably contains only classes which are in the image of the cohomology with compact supports. This kernel in general is nonempty. For instance,

THEOREM 2. $h^3(SL(3, \mathbf{Z})(7)) > h_1(T/SL(3, \mathbf{Z})(7)) + kh_1(X(7)) = 5815.$

Similar results could be obtained for other primes. The demonstration of this theorem depends upon the following.

PROPOSITION. *Let C be the cone of all $n \times n$ positive-definite symmetric matrices, A be the set of nonzero integral column vectors, and let $K = \{x \in C$: ${}^t a x a \geqslant 1$ for all a in $A\}$.*

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Let K_0 be the union of the compact faces of K. K_0 is $SL(n, \mathbf{Z})$-invariant under the action $(g, x) \mapsto {}^tg\,x\,g$, g in $SL(n, \mathbf{Z})$, x in C. If Γ is any torsion-free subgroup of finite index of $SL(n, \mathbf{Z})$ and $n \leqslant 4$, K_0/Γ is a deformation retract of C/Γ.

I do not know if this stays true for $n \geqslant 5$. The proof is similar to methods in [4] and [5].

The computation of K_0 for $n = 3$ is not difficult given some knowledge of 3-dimensional crystallography, and a description of K_0 for $n = 4$ has been graciously supplied to me by M. I. Štogrin. See also [6], [7].

In K_0, I have an explicit simplicial complex homotopic to C/Γ. Since C is contractible, I can use it to compute $H(\Gamma)$. I obtained Theorem 2 by decomposing the corresponding chain complex into $SL(3, \mathbf{Z}/7)$-invariant subspaces and taking the euler characteristics of invariant complexes, using [8].

For $n = 4$, this procedure is already too difficult to carry out by hand, but I can obtain one result:

THEOREM 3. *If Γ is as in the proposition above, the images of the $SL(4, \mathbf{R})$-invariant differential forms on $SL(4, \mathbf{R})/SO(4, \mathbf{R})$ are zero in*

$$\widetilde{H}^*(\Gamma\backslash SL(4, \mathbf{R})/SO(4, \mathbf{R}), \mathbf{R}),$$

thought of as de Rham Cohomology.

In view of [9], we can call Theorem 3 an "instability result".

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