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RUNGE FAMILIES AND INCREASING UNIONS OF STEIN SPACES

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The general question of whether an increasing union of Stein spaces is itself Stein has been open for some time. Recently J. Fornaess has produced a counterexample [1]. Here a complete answer is given.

It has been known that if each pair in the union form a Runge pair, then the question has an affirmative answer. However the union may be Stein without each pair being Runge, since it is also known that the conjecture is valid for Riemann domains spread over Stein manifolds.

On the other hand, the following generalization of the Runge concept to the entire family does characterize those families which answer the question affirmatively.

DEFINITION. An increasing family of analytic spaces $\cdots \subset X_n \subset X_{n+1} \subset \cdots$ in which each X_n is open in X_{n+1} is said to be a *Runge family* if for every compact $K \subset X_n$, there exists j such that $K \subset X_j$ and $\mathcal{O}(\bigcup X_n)$ uniformly approximates $\mathcal{O}(X_j)$ on K.

Then we have

THEOREM. *If $\cdots \subset X_n \subset X_{n+1} \subset \cdots$ is an increasing family of Stein spaces, each X_n open in X_{n+1}, then the following are logically equivalent*:

1. *$\bigcup X_n$ is Stein.*
2. *$\cdots \subset X_n \subset X_{n+1} \subset \cdots$ is a Runge family.*
3. *$H^1(\bigcup X_n, \mathcal{O}) = 0$.*

The proof will appear elsewhere, but the idea is to first note that $1 \Rightarrow 3$ trivially. Then the implication $3 \Rightarrow 2$ is proved using duality on complex spaces together with a Hahn-Banach style of argument. Finally $2 \Rightarrow 1$ is just a modification of the usual cohomology approximation theorem used to prove that the union is Stein when each pair is Runge.

Above it was mentioned that when the union is a Riemann domain over a Stein manifold, then the answer to the question is affirmative. This is usually proved via pseudoconvexity techniques. However, H. Laufer [2] has proved (essentially) that any increasing family of open sets in a Stein manifold is a Runge

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family. Thus the result for schlicht Riemann domains appears as a corollary of the theorem.

REFERENCES

1. J. E. Fornaess, *An increasing sequence of Stein manifolds whose limit is not Stein*, Math. Ann. (submitted).

2 H. B. Laufer, *On Serre duality and envelopes of holomorphy,* Trans. Amer. Math. Soc. **128** (1967), 414–436. MR **36** #5393.

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