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THE RIESZ DECOMPOSITION FOR VECTOR-VALUED AMARTS

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Let (Ω, F, P) be a probability space, $\mathbf{N} = \{1, 2, \ldots\}$, and let $(F_n)_{n\in\mathbf{N}}$ be an increasing sequence of σ-algebras contained in F. A *stopping time* is a mapping $\tau: \Omega \longrightarrow \mathbf{N} \cup \{\infty\}$, such that $\{\tau = n\} \in F_n$ for all $n \in \mathbf{N}$. The collection of bounded stopping times is denoted by T; under the natural ordering T is a directed set 'filtering to the right'.

Let $\mathbf{E}$ be a Banach space and consider a sequence $(X_n)_{n\in\mathbf{N}}$ of $\mathbf{E}$-valued random variables *adapted to* (F_n), i.e., such that $X_n: \Omega \longrightarrow \mathbf{E}$ is F_n-strongly measurable. EX (expectation of X) is the Pettis integral of X; $E_A X$ denotes $E(1_A \cdot X)$. The sequence (X_n) is called an *amart* iff each X_n is Pettis integrable and $\lim_T E(X_\tau)$ exists in the strong topology of $\mathbf{E}$.

The *real* Riesz decomposition theorem for amarts [4] asserts that an amart X_n can be uniquely written as a sum of a martingale Y_n, and an amart Z_n that converges to zero in nearly all possible ways: $Z_n \longrightarrow 0$ a.e. and in L^1, and $Z_\tau \longrightarrow 0$ in L^1.

As a consequence of this result, and of the real amart convergence theorem [1]—the first important result involving discrete parameter amarts—we obtain

THEOREM 1. *Let $\mathbf{E} = \mathbf{R}$. If (X_n, F_n) is an amart, then (and only then) for each increasing sequence $\tau_n \geq n$ in T, $E^{F_n}X_{\tau_n} - X_n \longrightarrow 0$ a.e. and in L^1.*

The Banach-valued Riesz decomposition is the main result of the present note. The *Pettis norm* of a random variable X is $\|X\| = \sup E|f(X)|$ where the supremum is over all $f \in \mathbf{E}'$ with $|f| \leq 1$ [6].

A *potential* is an amart that converges to zero in the Pettis norm. A sequence of adapted random variables is said to be *of class* (B) iff $\sup_T E|X_\tau| < \infty$. We prove

THEOREM 2 (RIESZ DECOMPOSITION). *Let $\mathbf{E}$ be a Banach space with the Radon-Nikodym property and let (X_n, F_n) be an $\mathbf{E}$-valued amart such that*

$$\liminf E|X_n| < \infty. \tag{1}$$

(i) *X_n can be uniquely written as the sum of a martingale Y_n and a*

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potential Z_n. $(Z_\tau)_{\tau\in T}$ *converges to zero in Pettis norm.*

(ii) *If* $\mathbf{E}'$ *is separable and* $(X_n, \mathcal{F}_n)$ *is of class* (B), *then* $Z_n \longrightarrow 0$ *a.e. weakly.*

SKETCH OF PROOF. (Complete proof will appear elsewhere). (i) For each $A \in \mathcal{F}_m$, $\lim E_A X_n = \mu_m(A)$ exists (cf. [3]). (1) implies that μ_m has finite variation. $Y_m = d\mu_m/dP$ is a martingale and $Z_m = X_m - Y_m$ satisfies $E_A Z_m \longrightarrow 0 \; \forall A \in \mathcal{F}_m$. Let $\epsilon_i \downarrow 0$. For each m choose $A_m \in \mathcal{F}_m$ so that

$$\sup_{A\in \mathcal{F}_m} |E_A Z_m| - |E_{A_m} Z_m| < \epsilon_m. \tag{2}$$

We can find an integer $n_m > m$ such that $|E_{A_m^c} Z_{n_m}| < \epsilon_m$. Define a stopping time τ_m by $\tau_m = m$ on A_m; $\tau_m = n_m$ on A_m^c. Then for each m, $|E_{A_m} Z_m - EZ_{\tau_m}| = |E_{A_m^c} Z_{n_m}| < \epsilon_m$. Since Z_n is an amart, $\lim_m EZ_{\tau_m} = \lim_m EZ_m = 0$. It follows that $E_{A_m} Z_m \longrightarrow 0$; hence, by (2), $\sup_{A\in\mathcal{F}_m} |E_A Z_m| \longrightarrow 0$ which implies that $\|Z_m\| \longrightarrow 0$.

For each increasing sequence of bounded stopping times τ_n, $(Z_{\tau_n})_{n\in\mathbf{N}}$ is an amart with respect to $(\mathcal{F}_{\tau_m})_{m\in\mathbf{N}}$ [4]. Therefore $\|Z_{\tau_m}\| \longrightarrow 0$; it follows that $\lim_T \|Z_\tau\| = 0$.

The proof of (ii) uses the vector amart convergence theorem [3].

In the discussing of examples relevant to Theorem 2, the following result is useful: For any $\mathbf{E}$, any amart, the Riesz decomposition holds if and only if $d\mu_m/dP$ exists for each m.

EXAMPLE 1. The assumption that $\mathbf{E}$ has the Radon-Nikodym property cannot be omitted. Let $\{e_n^i,\ n \in \mathbf{N},\ 1 \leqslant i \leqslant 2^n\}$ be the standard basis for the Banach space c_0 (in any order). Let $A_n^i \cap A_n^j = \emptyset$ if $i \neq j$, and $P(A_n^i) = 2^{-n}$. Let

$$X_n = \sum_{k=1}^{n} \sum_{i=1}^{2^k} e_k^i 1_{A_k^i}.$$

Let $\mathcal{F}_n = \mathcal{F}, n = 1, 2, \ldots.$ $(X_n, \mathcal{F}_n)$ is a bounded amart (cf. [3]), but the Riesz decomposition fails.

EXAMPLE 2. Assumption (1) cannot be omitted. Let (e_i) be the usual basis of $\mathbf{E} = l_p$ for some p, $1 \leqslant p < 2$. Let Z_n be independent real random variables with $P(Z_n = 1) = P(Z_n = -1) = ½$. Set $X_n = \Sigma_{k=1}^n a_k e_k Z_k$ where a_k are positive constants with $\Sigma_k a_k^p = \infty, \Sigma a_k^2 < \infty$. Then $\lim E|X_n| = \infty$, and one proves that X_n is an amart with respect to the constant sequence of σ-algebras $\mathcal{F}$, that does not have the Riesz decomposition.

The first example in [3] (or [4]) shows that a bounded potential in a Hilbert space need not converge a.e. strongly, or in L_E^1 norm; the second example shows that an L^1 bounded potential not of class (B) need not converge a.e. weakly.

REFERENCES

1. D. G. Austin, G. A. Edgar and A. Ionescu Tulcea, *Pointwise convergence in terms of expectations*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **30** (1974), 17–26. MR 50 #11402.

2. A. Bellow, *On vector-valued asymptotic martingales*, Proc. Nat. Acad. Sci. U.S.A. (to appear).

3. R. V. Chacon and L. Sucheston, *On convergence of vector-valued asymptotic martingales*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **33** (1975), 55–59.

4. G. A. Edgar and L. Sucheston, *Amarts: A class of asymptotic martingales*, J. Multivariate Analysis (to appear).

5. ———, *Les amarts: une classe de martingales asymptotiques*, C. R. Acad. Sci. Paris (to appear).

6. B. J. Pettis, *On integration in vector spaces*, Trans. Amer. Math. Soc. **44** (1938), 277–304.

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