BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 4, July 1976

A GRADIENT ESTIMATE AT THE BOUNDARY FOR SOLUTIONS OF QUASILINEAR ELLIPTIC EQUATIONS

BY RONALD GARIEPY AND WILLIAM P. ZIEMER[1]

Communicated by Hans F. Weinberger, April 8, 1976

1. **Introduction.** The purpose of this note is to present a result concerning regularity at the boundary of bounded, weak solutions of equations of the form

$$\operatorname{div} A(x, u, u_x) = B(x, u, u_x) \tag{1}$$

where A and B are, respectively, vector and scalar valued Baire functions defined on $\Omega \times R^1 \times R^n$ that satisfy

$$\begin{aligned} &|A(x, u, w)| \leqslant a_0|w|^{p-1} + a_1|u|^{p-1} + a_2, \\ &|B(x, u, w)| \leqslant b_0|w|^p + b_1|w|^{p-1} + b_2|u|^{p-1} + b_3, \\ &w \cdot A(x, u, w) \geqslant c_0|w|^p - c_1|u|^p - c_2. \end{aligned} \tag{2}$$

Here, Ω is an open subset of R^n, $1 < p < n$, $c_0 > 0$, $a_0 \geqslant 0$, $b_0 \geqslant 0$, and the remaining coefficients are nonnegative, measurable functions in the respective Lebesgue classes

$$\begin{aligned} &a_1, a_2 \in L_{n/p-1}(\Omega), \qquad b_1 \in L_{n/1-\delta}(\Omega), \\ &c_1, c_2, b_2, b_3 \in L_{n/p-\delta}(\Omega) \quad \text{where } 0 < \delta < 1. \end{aligned}$$

A weak solution of (1) is a function u in the Sobolev space $W_p^1(\Omega)$ that satisfies $\int_\Omega A \cdot \nabla\varphi + B \cdot \varphi = 0$ whenever φ is a smooth function with compact support in Ω.

It has been shown in [LU], [S], and [T] that a weak solution of (1) is Hölder continuous on compact subsets of Ω. In connection with boundary regularity, it was established in [LSW] that a point $x_0 \in \partial\Omega$ is regular for solutions of linear, uniformly elliptic equations in divergence form with bounded, measurable coefficients if and only if x_0 is regular for Laplace's equation. Later, Stampacchia [ST] extended this result to a wider class of linear elliptic equations. For solutions of quasilinear equations of the form $\operatorname{div} A(x, u_x) = 0$, but subject to conditions more restrictive than (2), Maz'ya [M] established regularity at $x_0 \in \partial\Omega$ provided the following condition is satisfied:

AMS (MOS) subject classifications (1970). Primary 35D10, 35B45; Secondary 35J25, 35J60, 35J67.

[1] Research supported in part by a grant from the National Science Foundation.

$$\int_0^1 [\Gamma_p(B(x_0, r) - \Omega)r^{p-n}]^{1/(p-1)} \frac{dr}{r} = \infty. \tag{3}$$

Here $B(x_0, r)$ denotes the n-ball of radius r centered at x_0 and Γ_p is a capacity defined on all sets $E \subset R^n$ by

$$\Gamma_p(E) = \inf\left\{\int |\nabla f|^p\right\}$$

where the infimum is taken over all $f \in L_{np/(n-p)} \cap W_p^1(R^n)$ for which $E \subset \operatorname{int}\{x: f(x) \geqslant 1\}$. In view of the fact that Γ_2 is Newtonian capacity, one observes that (3) is precisely the classical Wiener condition when $p = 2$.

2. **The main results.** Given a continuous function f on $\partial\Omega$ and $x_0 \in \partial\Omega$, we will say $u(x_0) \leqslant f(x_0)$ *weakly* for functions $u \in W_p^1(\Omega)$ provided that whenever η is a smooth function supported in $B(x_0, r)$ and $f < k$ in $B(x_0, r) \cap \partial\Omega$, then $(u - k)^+\eta \in W_{p,0}^1(\Omega)$. A similar definition is given for $u(x_0) \geqslant f(x_0)$ *weakly* and, therefore, we can give meaning to $u(x_0) = f(x_0)$ *weakly.*

As a direct consequence of the gradient estimate below, (5), we obtain the following

THEOREM. *Suppose f is a continuous function on $\partial\Omega$ and let $u \in W_p^1(\Omega)$ be a bounded weak solution of* (1) *such that $u(x_0) = f(x_0)$ weakly. If* (3) *holds and*

$$\int_0^1 \|a_1 + a_2\|_{n/(p-1),B(x_0,r)}^{1/(p-1)} \frac{dr}{r} < \infty, \tag{4}$$

then $\lim_{x\to x_0; x\in\Omega} u(x) = f(x_0)$.

The notation in (4) indicates the norm of $a_1 + a_2$ taken relative to the n-ball $B(x_0, r)$. Of course, if it is assumed that $a_1, a_2 \in L_q$ where $q > n/(p - 1)$, then clearly (4) is satisfied.

It is interesting to observe that the regularity results of **[LSW]**, **[ST]**, and **[M]** are obtained by employing potential-theoretic techniques, whereas ours is based primarily on information obtained from the differential equation itself. Indeed, the following estimate is the vital component.

If $u \in W_p^1(\Omega)$ is a bounded, weak solution of (1), let $\mu(r) = \sup\{u(x): x \in B(x_0, r)\}$, where $x_0 \in \partial\Omega$. Suppose $k > f(x_0)$ and let $u_k = (u - k)^+$.

THEOREM. *There is a constant C depending only on n, p, the bound for u, the coefficients in* (2), *and δ such that for all sufficiently small r,*

$$r^{p-n}\int_{B(x_0,r/4)} |\nabla u_k|^p \leqslant C[\mu(2r) - \mu(r) + a(r)]^{p-1} \tag{5}$$

whenever $u(x_0) \leqslant f(x_0)$ weakly and where

$$a(r) = r + \|a_1 + a_2\|^{1/(p-1)}_{n/(p-1),B(x_0,r)}$$
$$+ \|c_1 + c_2 + b_2 + b_3\|^{1/(p-\delta/2)}_{n/(p-\delta/2),B(x_0,r)}.$$

Suppose u is a solution of (1) such that $u(x_0) = f(x_0)$ weakly but that $\lim_{x\to x_0; x\in\Omega} u(x) \neq f(x_0)$. If (4) holds the gradient estimate (5) is used to show that there is a set E which is Γ_p-thin at x_0 (see **[ME]** for definition) such that $u(x)$ tends to a limit as $x \longrightarrow x_0$, $x \notin E$. Thus, in terms analogous to the classical case, u has a Γ_p-fine limit at x_0.

Proofs of these and other results will appear elsewhere.

BIBLIOGRAPHY

[LSW] W. Littman, G. Stampacchia and H. F. Weinberger, *Regular points for elliptic equations with discontinuous coefficients,* Ann. Scuola Norm. Sup. Pisa (3) **17** (1963), 43–77. MR **28** #4228.

[LU] O. A. Ladyženskaja and N. N. Ural'ceva, *Linear and quasilinear elliptic equations,* "Nauka", Moscow, 1964; English transl., Academic Press, New York, 1968. MR **35** #1955; **39** #5941.

[M] V. G. Maz'ja, *On the continuity at a boundary point of the solutions of quasilinear elliptic equations,* Vestnik Leningrad. Univ. **25** (1970), no. 13, 42–55. (Russian) MR **43** #706.

[ME] N. Meyers, *Continuity property of potentials,* Duke Math. J. **42** (1975), 157–166.

[ST] G. Stampacchia, *Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus,* Ann. Inst. Fourier (Grenoble) **15** (1965), fasc. 1, 189–258. MR **33** #404.

[S] J. B. Serrin, Jr., *Local behavior of solutions of quasi-linear equations,* Acta Math. **111** (1964), 247–302. MR **30** #337.

[T] N. S. Trudinger, *On Harnack type inequalities and their application to quasilinear elliptic equations,* Comm. Pure Appl. Math. **20** (1967), 721–747. MR **37** #1788.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KENTUCKY, LEXINGTON, KENTUCKY 40506

DEPARTMENT OF MATHEMATICS, INDIANA UNIVERSITY, BLOOMINGTON, INDIANA 47401