BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 4, July 1976

SPECTRAL CLASSIFICATION OF OPERATORS AND OPERATOR FUNCTIONS

BY I. GOHBERG,[1] M. A. KAASHOEK AND D. C. LAY

Communicated by Paul R. Halmos, January 23, 1976

Let A and B be holomorphic functions from an open set Ω in the complex plane into the Banach space $L(X, Y)$ of all bounded linear operators between two Banach spaces X and Y. The functions A and B are called *equivalent on* Ω (see [3]) if there exist holomorphic operator functions E and F on Ω, whose values are bijective bounded linear operators on X and Y, respectively, such that

$$A(\lambda) = F(\lambda)B(\lambda)E(\lambda), \quad \lambda \in \Omega.$$

The concept of equivalence is also of interest in the case that ($X = Y$ and) A and B are linear functions of the form $C - \lambda I$. In that case it provides a language in which the spectral structure of a linear operator at a point may be classified. Two operators T and S are said to have the same *spectral structure* at a point λ_0 if the operator functions $T - \lambda I$ and $S - \lambda I$ are equivalent on an open neighbourhood of λ_0. More generally, we say that two operator functions A and B belong to the same *spectral class* at a point λ_0 if there exists an open neighbourhood of λ_0 on which A and B are equivalent.

Let $D(\lambda) = (\lambda - \lambda_0)^{k_1}P_1 + \cdots + (\lambda - \lambda_0)^{k_n}P_n + P_0$, where $k_1, \ldots, k_n$ are positive integers, $P_1, \ldots, P_n$ are mutually disjoint one dimensional projections and $P_0 = I - (P_1 + \cdots + P_n)$. An operator function A belongs to the spectral class generated by D at λ_0 if and only if $A(\lambda)$ is bijective for λ near λ_0, $A(\lambda_0)$ is Fredholm and the partial multiplicities of A at λ_0 are given by the numbers $k_1, \ldots, k_n$ (see [5]). Other examples of spectral classes can be found in [1] and [7].

The problem of finding the simplest representative of a spectral class containing a given operator function A has many variants. One possibility is to look for functions of the form $T - \lambda I$. It is clear that in this way the problem is not always solvable. However after a suitable extension of the operator function A and the underlying spaces the problem has a positive solution.

If Z is a Banach space, the *Z-extension* of A is the operator function whose value at λ is the operator $A(\lambda) \oplus I_Z$ in $L(X \oplus Z, Y \oplus Z)$, i.e., the direct sum of

AMS (MOS) subject classifications (1970). Primary 47A10; Secondary 47A20, 47A50, 47B30, 30A96.

Key words and phrases. Spectral classification of operators and operator functions, Fredholm operators, equivalence, extension, linearization.

[1]Partially supported by an NSF grant.

$A(\lambda)$ in $L(X, Y)$ and the identity operator I_Z on Z.

THEOREM 1. *Let Ω be a domain in* C *whose boundary is the union of a finite number of nonintersecting Jordan curves, and let A be an operator function, holomorphic on Ω and continuous on the closure $\bar{\Omega}$, with values in $L(X, Y)$. Then there exists a Banach space Z such that the Z-extension of A is equivalent on Ω to a linear bundle $T - \lambda I$.*

In the case $X = Y$ the operator T in Theorem 1 can be chosen to be the operator on the space of all X-valued continuous functions on the boundary Γ of Ω defined by

$$(Tf)(z) = zf(z) - (2\pi i)^{-1} \int_\Gamma [I - A(\zeta)]f(\zeta)d\zeta.$$

Here we assume 0 to be a point in Ω.

The next theorem is helpful in finding the simplest representative in a given spectral class. By definition the *singular set* of an operator function B is the set of all λ in the domain of B such that $B(\lambda)$ is not bijective.

THEOREM 2. *Let Ω and A be as in Theorem* 1, *except that A has values in $L(X)$. Suppose that $A(\cdot) = A_1(\cdot)A_2(\cdot) \cdots A_n(\cdot)$, where each A_j is holomorphic on Ω and continuous on $\bar{\Omega}$, with values in $L(X)$. Suppose that the singular sets of $A_1, \ldots, A_n$ are pairwise disjoint compact subsets of Ω. Then the X^{n-1}-extension of A is equivalent on Ω to the function on $X \oplus \cdots \oplus X$ whose value at λ is the direct sum operator $A_1(\lambda) \oplus \cdots \oplus A_n(\lambda)$.*

The proof of Theorem 2 is based on the next theorem, which in turn is proved by using Theorem 1 and a result of [6] (see also [2]).

THEOREM 3. *Let Ω be a domain in* C *whose boundary is the union of a finite number of nonintersecting Jordan curves. Let $A, B: \Omega \longrightarrow L(X)$ be holomorphic on Ω and continuous on $\bar{\Omega}$, and suppose that the singular sets of A and B are disjoint compact subsets of Ω. Then given a holomorphic function $C: \Omega \longrightarrow L(X)$, there exist holomorphic functions $Z, W: \Omega \longrightarrow L(X)$, such that $A(\lambda)Z(\lambda) + W(\lambda)B(\lambda) = C(\lambda), \lambda \in \Omega$.*

With regard to the definition of equivalence several other problems can be mentioned. First of all there is the problem about the connection between "global" equivalence on Ω and "local" equivalence on a neighbourhood of each point in Ω. In general the two types of equivalence are not the same. This follows from a counterexample in [4, §10]. (This counterexample deals with a holomorphic operator function $P(\cdot)$, defined on a nonsimply connected domain Ω, whose values are projections of a Banach space with a nonconnected general linear group. This function $P(\cdot)$ is equivalent to the same fixed projection Q on a neighbourhood of each point of Ω, but not on the whole of Ω.) Recently

J. Leiterer has constructed an example of a simply connected domain Ω and two holomorphic operator functions A and B on Ω, with values in a separable Hilbert space such that A and B are locally equivalent at each point of Ω, and, in addition, there exists an infinitely differentiable function C on Ω whose values are bijective operators such that $A(\lambda) = C(\lambda)B(\lambda)$ for $\lambda \in \Omega$. However A and B are not globally equivalent on Ω.

After these two examples the following problem remains open: Under what conditions does "local" equivalence on a neighbourhood of each point in Ω imply "global" equivalence on Ω. This problem is of particular interest if the operator functions A and B are of the form $A(\lambda) = T - \lambda I$ and $B(\lambda) = S - \lambda I$. Another problem which is still open is the question whether equivalence of $T - \lambda I$ and $S - \lambda I$ on a sufficiently large open disc implies similarity of T and S. The last two problems have positive solutions in the finite dimensional case and some other special cases.

REFERENCES

1. H. Bart, M. A. Kaashoek and D. C. Lay, *Relative inverses of meromorphic operator functions*, Univ. of Maryland Tech. Report # TR 74–71, 1974.

2. Ju. L. Daleckiĭ and M. G. Kreĭn, *Stability of solutions of differential equations in Banach space*, "Nauka", Moscow, 1970; English transl., Transl. Math. Monographs, vol. 43, Amer. Math. Soc., Providence, R. I., 1975. MR **50** # 5125, 5126.

3. I. C. Gohberg, *On some topics of spectral theory of finite-meromorphic operator functions*, Izv. Akad. Nauk Armjan S.S.R. **6** (1971), 160–181 (Russian).

4. I. C. Gohberg and J. Leiterer, *Cocycles, operator-valued functions and families of subspaces*, Mat. Issled **8** (1973), vyp. 2 (28), 23–56, 188 (Russian). MR **49** # 1121.

5. I. C. Gohberg and E. I. Sigal, *An operator generalization of the logarithmic residue theorem and the theorem of Rouché*, Mat. Sbornik **84 (126)** (1971), 607–629 = Math. USSR Sbornik **13** (1971), 603–625. MR **47** # 2409.

6. M. Rosenblum, *On the operator equation $BX - XA = Q$*, Duke Math. J. **23** (1956), 263–269. MR **18**, 54.

7. E. I. Sigal, *Factor multiplicity of the characteristic number of a meromorphic operator function*, Mat. Issled. **5** (1970), vyp. 4 (18), 136–152. (Russian) MR **44** # 2073.

DEPARTMENT OF MATHEMATICS, TEL-AVIV UNIVERSITY, RAMAT-AVIV, TEL-AVIV, ISRAEL

WISKUNDIG SEMINARIUM, VRIJE UNIVERSITEIT, AMSTERDAM-11, THE NETHERLANDS

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARYLAND, COLLEGE PARK, MARYLAND 20742