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DEFORMATIONS OF GEODESIC FIELDS

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We describe here a deformation theorem for geodesic fields on a Riemannian manifold and an obstruction whose vanishing is necessary and sufficient for deforming one geodesic field into another. As an application we prove that every smooth manifold of dimension $\geqslant 2$ can be given a Riemannian metric with a nontriangulable cut locus. In [5] we obtain an "equivariant" deformation theorem by entirely different methods, and a consequent strengthening of the cut locus results. We thank Professors Richard Hamilton and Albert Nijenhuis for insights gained in conversation with them.

1. **The cut locus.** On each geodesic from the point p on the compact Riemannian manifold M, the *cut point* is the last point to which the geodesic minimizes distance, and the *cut locus* $C(p)$ is the set of these. This notion was introduced by Poincaré in 1905 and has since played an important role in global differential geometry [3].

In 1935 Myers [4] showed that on a compact analytic surface, the cut locus can be triangulated as a finite graph. Recently this has been extended to arbitrary dimensions by the work of Buchner, Mather, Hironaka and Kato [2]. Buchner proved in his thesis [1] that on any smooth compact manifold of dimension $\leqslant 5$, there is an open and dense set of Riemannian metrics for which the cut locus of a point p is structurally stable under perturbation of metric (and one expects triangulable). By contrast we prove

THEOREM 1. *On any smooth manifold of dimension* $\geqslant 2$, *there is a Riemannian metric and a point* p *with nontriangulable cut locus* $C(p)$.

2. **The connection with deformations of geodesic fields.** The problem of preassigning a cut locus led us to the study of deformations of geodesic fields. In Figure 1 we start with a round sphere and try to preassign a cut locus C of the south pole. We draw out from C a smooth family G' of geodesics heading roughly south, and from the south pole the family G of geodesics heading north. It seems natural to try to deform the round metric in a neighborhood of the equator so as to deflect the geodesic field G' into G, like a lens focusing some light rays from C to make them converge at the south pole.

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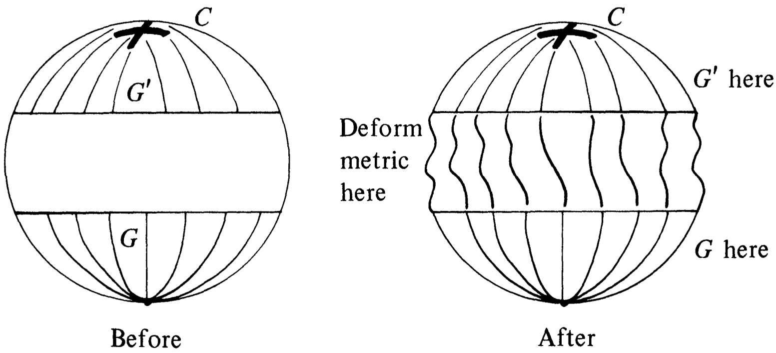

FIGURE 1

3. **The general deformation theorem.** In Figure 2, the round sphere has been replaced by a Riemannian manifold N, and the equator by a compact hypersurface M which separates N. Two geodesic fields G and G' are given on N, *both* crossing M transversely. PROBLEM. Is it possible to deform the metric on N near M so as to produce a field of geodesics connecting G below M to G' above? To

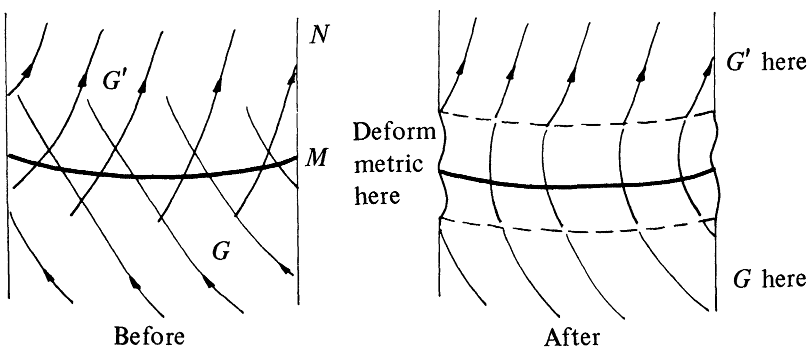

FIGURE 2

state the result, let V denote the unit tangent vector field to G and ω the dual one-form on N defined by $\omega(U) = U \cdot V$, and similarly define V' and ω' relative to G'.

THEOREM 2. *A necessary and sufficient condition for being able to deform the metric on N near M so as to produce a field of geodesics connecting G and G', is the existence of a diffeomorphism ϕ: $M \longrightarrow M$, concordant to the identity, such that the one-form on M, $\omega|_M - \phi^*(\omega'|_M)$, is exact.*

The map ϕ tells which geodesics of G and G' are hooked together.

4. **Integrable geodesic fields.** A field G of geodesics on N^n is *integrable* if the orthogonal $(n-1)$ plane distribution is integrable in the usual sense of being tangent to a foliation. In this case $d\omega = 0$, so that ω determines a cohomology class $[\omega] \in H^1(N; R)$, any by restriction $[\omega|_M] \in H^1(M; R)$. Then Theorem 2 simplifies to

THEOREM 3. *A necessary and sufficient condition for being able to deform the metric on N near M so as to produce a field of geodesics connecting the integrable geodesic fields G and G′, is that*

$$[\omega|_M] = [\omega'|_M] \in H^1(M; R).$$

This is sufficient to prove Theorem 1, since geodesics from a fixed point p form an integrable field wherever they do not cross.

ADDENDUM TO THEOREM 3. *The deformed metric on N can always be chosen pointwise conformal to the original metric.*

This is of use in anticipated applications to optics.

5. Nontriangulable cut loci. Refer back to Figure 1. It is not difficult to choose a field G' of geodesics coalescing along a set C which is disconnected into infinitely many components by removal of the north pole and is therefore nontriangulable. At the same time we arrange to satisfy the conditions of Theorem 3, which then yields a metric on the sphere having C as cut locus of the south pole. Albert Nijenhuis pointed out to us that such a metric can be seen in three-space on a sphere, round except for an infinite sequence of bumps of decreasing size, along the equator.

To obtain a metric with nontriangulable cut locus on M^m, $m \geqslant 2$, begin with one on S^m as above. Keeping the sphere round except near the eastern half of the equator, attach a copy of M^m to the western hemisphere by a tube as in Figure 3. No geodesic on the union $M^m \# S^m$ which starts from the south

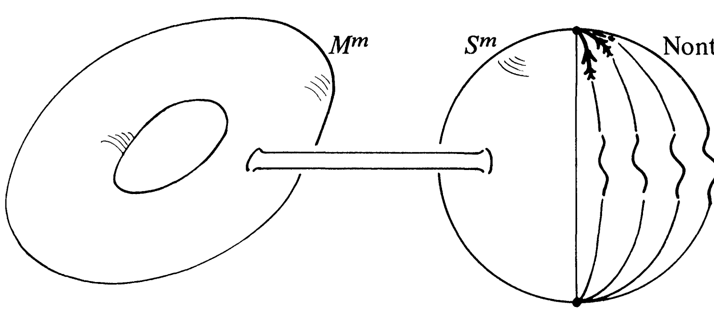

FIGURE 3

pole and heads into the open western hemisphere can have its cut point in the eastern hemisphere, save at the north pole. This guarantees that the cut locus of the south pole on $M^m \# S^m$ is still disconnected into infinitely many components by removal of the north pole, and is hence nontriangulable, proving Theorem 1.

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