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A LANGUAGE FOR TOPOLOGICAL STRUCTURES WHICH SATISFIES A LINDSTRÖM-THEOREM

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I. **The language** L^t. Let L_2 be the 2-sorted first order language appropriate for structures $(\mathfrak{A}, \alpha, \epsilon)$, where $\mathfrak{A}$ is a L-structure and α is a set of subsets of A. We call $(\mathfrak{A}, \alpha)$ topological if α is a topology. We call a formula of L_2 topological if it is built up using the set quantifier $\exists X$ only in the form $\exists X(t \in X \wedge \phi)$, X does not occur positively in ϕ. (X occurs positively in ϕ if a free occurrence of X in ϕ is inside the scope of an even number of negation symbols. *Note*. Primitive symbols are $\wedge, \neg, \exists x, \exists X$.) L^t is defined as the set of topological sentences of L_2.[1]

LEMMA. (a) *Define* $\tilde{\beta} = \{\bigcup s | s \subset \beta\}$. *Then for all* $\phi \in L^t$, $(\mathfrak{A}, \beta) \models \phi$ *iff* $(\mathfrak{A}, \tilde{\beta}) \models \phi$. (*I.e.* ϕ *is invariant in the sense of Garavaglia* [1].)

(b) $\tilde{\beta}$ *is a topology iff* $(\mathfrak{A}, \beta) \models$ top, *where* top *is the* L^t*-sentence*

$$\forall x(\exists X \wedge x \in X) \wedge \forall x \forall X(x \in X \longrightarrow \forall Y(x \in Y \longrightarrow \exists Z(x \in Z \wedge \forall y(y \in Z \longrightarrow y \in X \wedge y \in Y)))).$$

In the sequel "model" means "topological model".

COROLLARY (see [1]). (a) $T \subset L^t$ *has a model iff* $T \cup \{\text{top}\}$ *is consistent* (*in the* 2*-sorted predicate calculus of* L_2).

(b) *The set of* L^t*-sentences true in all models is r.e.*

(c) L^t *satisfies the compactness theorem*: T *has a model iff every finite subset of* T *has a model.*

(d) L^t *satisfies the downward Löwenheim-Skolem theorem* (L *countable*): *If* T *has an infinite model, it has a "countable" model* $(\mathfrak{A}, \alpha)$, *i.e.* A *countable*, α *having a countable base.*

By the methods of the next section we can prove a Lindström-

THEOREM. *Let* L^* *be a language for topological structures extending* L^t *and satisfying the compactness theorem and the downward Löwenheim-Skolem theorem. Then* $L^* = L^t$.

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[1] I have seen, that L^t was first considered for topological spaces by T. A. McKee in two articles in the Z. Math. Logik Grundlagen Math. **21** (1975), 405-408 and ibid. (1976).

COROLLARY. *$\phi \in L_2$ is invariant iff* top$\vdash \phi \longleftrightarrow \psi$ *for a $\psi \in L^t$.*

So L^t seems to be the natural language for topological structures.

REMARK. One can translate the (weaker) topological logics $L(Q^n)$, $L(I^n)$ considered in [2], [3] into L^t. "f continuous" and the separation axioms T_0 – T_3 are expressible in L^t.

II. **The Ehrenfeucht-Fraissé game for L^t.** There are two players, I and II, for the ν-E.F. game between the two models $\mathfrak{M}_i = (\mathfrak{A}_i, \alpha_i)$, $i \in 2$. The kth move is: I chooses $i \in 2$, $a_i \in A_i$ and a neighbourhood N_i^k of a_i^k. Then II chooses $a_{1-i}^k \in A_{1-i}$ and a neighbourhood N_{1-i}^k of a_{1-i}^k. After ν moves II has won if $\{\langle a_0^k, a_1^k\rangle \mid < \nu\}$ is a partial isomorphism between $\mathfrak{A}_0$ and $\mathfrak{A}_1$ and if for all N_i^k chosen by I and all $j < \nu$, $a_{1-i}^j \in N_{1-i}^k \Rightarrow a_i^j \in N_i^k$.

THEOREM. (a) *$\mathfrak{M}_0$ and $\mathfrak{M}_1$ are L^t-elementarily equivalent iff player* II *has a winning strategy for all n-E.F.-games between $\mathfrak{M}_0$ and $\mathfrak{M}_1$, $n \in \omega$.*

(b) *Suppose* $|A_i| \leqslant \kappa$, and *α_i possesses a base of power $\leqslant \kappa$, $i \in 2$. Then $\mathfrak{M}_0 \cong \mathfrak{M}_1$ iff there is a winning strategy for* II *in the κ-E.F.-game between $\mathfrak{M}_0$ and $\mathfrak{M}_1$ (L finite).*

REMARK. The E.F.-game described above approximates isomorphisms. One can design E.F.-games approximating other relations between models: e.g. "$\mathfrak{M}_0 \subset \mathfrak{M}_1$" (i.e. $\mathfrak{A}_0$ is a substructure of $\mathfrak{A}_1$, α_0 the subspace topology), "$\mathfrak{A}_0 = \mathfrak{A}_1$, α_0 coarser than α_1" or "$\mathfrak{M}_0$ is a continuous and homomorphic image of $\mathfrak{M}_1$".

III. **Saturation.** We call $(\mathfrak{A}, \alpha)$ (κ-) saturated if there is a base β for α s.t. $(\mathfrak{A}, \beta)$ is (κ-) saturated in the usual sense of L_2.

THEOREM. (a) *Every L^t-theory has a κ-saturated model.*

(b) *If U is a κ-good ultrafilter on I and $|L| < \kappa$, then $\mathfrak{M}^I/U$ is κ-saturated* $((\mathfrak{A}, \alpha)^I/U = (\mathfrak{A}^I/U, \widetilde{(\alpha^I/U)}))$.

(c) *Two saturated, L^t-elementarily equivalent models of the same cardinality are isomorphic.*

(d) *Two models are L^t-elementarily equivalent iff they have isomorphic ultrapowers.*

IV. **Definability.** $T \vdash_t \phi$ means: ϕ holds in all models of T.

By either the methods of II or III we can prove the

INTERPOLATION THEOREM. *Assume $\vdash_t \phi \longrightarrow \psi$ and $\phi, \psi \in L^t$. Then there is a $\theta \in L^t$ containing only nonlogical symbols which occur in both ϕ and ψ, s.t. $\vdash_t \phi \longrightarrow \theta$ and $\vdash_t \theta \longrightarrow \psi$.*

REMARKS. (a) *The analogue of Lyndon's interpolation theorem holds.*

(b) *Beth's theorem follows as usual for L^t.*

Define ϕ to be universal if ϕ becomes universal in the usual sense if we erase all set-quantifiers. We have the

PRESERVATION THEOREM. *Let* $T \subset L^t$, $\phi, \psi \in L^t$. *The following are equivalent*: (a) *For all T-models* $\mathfrak{M}_0 \supset \mathfrak{M}_1$, $\mathfrak{M}_0 \models \phi \Rightarrow \mathfrak{M}_1 \models \psi$.

(b) *There is a universal* $\theta \in L^t$ *s.t.* $T \vdash_t \phi \longrightarrow \theta$, $T \vdash_t \theta \longrightarrow \psi$.

REMARK. There are syntactical characterisations of the L^t-sentences preserved by various other relations between models (e.g. the relations given in (II)).

V. **Decidability**. The relation "x cannot be separated from y by disjoint closed neighbourhoods" can be arbitrary in T_2-spaces. So we have

THEOREM ($L = \varnothing$). *The theory of* T_2*-spaces is undecidable.*

The countable T_3-spaces have a base of clopen sets. Using this and the decidability of the monadic theory of countable trees we can prove

THEOREM (L CONSISTS OF UNARY PREDICATES). *The theory of* T_3*-spaces with a finite number of distinguished subsets is decidable.*

REMARK. The L^t-elementary types can be characterized by simple invariants, e.g. all T_3-spaces without isolated points are elementarily equivalent. In fact this theory is "$\aleph_0$-categorical".

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