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MARKOV PROCESSES ON MANIFOLDS OF MAPS

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1. **Introduction.** In this note we describe a construction of a Markov process on a manifold of maps starting from a Gaussian measure on the space of sections of an associated vector bundle. Let S be a compact metric space of finite metric dimension and M a smooth complete finite dimensional Riemannian manifold. Our basic construction gives a family $\{\nu_t: t \geqslant 0\}$ of Borel probability measures on the space $C(S \times M, M)$ of continuous functions from $S \times M$ to M with the compact-open topology. The multiplication $(f, g)(s, m) = f(s, g(s, m))$ for $f, g \in C(S \times M, M)$ makes $C(S \times M, M)$ into a topological semigroup with identity. Then $\nu_t * \nu_s = \nu_{t+s}$ for $s, t \geqslant 0$ and the right translates of the ν_t give transition probabilities for a Markov process on $C(S \times M, M)$ with continuous sample paths. The left action of $C(S \times M, M)$ on $C(S, M)$ induces a Markov process on $C(S, M)$ with transition probability $\nu_{t,g}$ = image of ν_t under the action of $C(S \times M, M)$ on $g \in C(S, M)$.

2. **Statement of results.** Let ξ denote the product bundle $S \times TM \longrightarrow S \times M$ and $C(\xi)$ the space of continuous sections of ξ. Given a Gaussian measure μ of mean zero on $C(\xi)$, define

$$Q(s, x, t, y) = \int f(s, x) \otimes f(t, y)\, d\mu(f) \in T_xM \otimes T_yM \quad \text{for all } s, t \in S,\ x, y \in M.$$

Q is a reproducing kernel for the bundle ξ (see Baxendale [1]) and determines μ uniquely. Let $X \in C(\xi)$.

For a closed isometric embedding of M inside some Euclidean space V, let $h(x)$ denote the second fundamental form for $M \subset V$ at $x \in M$. Using the natural inclusion $T_xM \subset V$ and orthogonal projection $V \longrightarrow T_xM$, we think of X, Q and h taking values in V and its various tensor products.

THEOREM 1. *Suppose there exists a closed isometric embedding $M \subset V$ such that* (i) *h is bounded and uniformly Lipschitz with respect to the metric on M induced from V.*

Suppose moreover that there exist a Gaussian measure μ on $C(\xi)$, $X \in C(\xi)$ and $\alpha > 0$, $C > 0$ such that

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(ii)
$$\operatorname{tr}(Q(s, x, s, x)) \leqslant C, \quad \forall s, x,$$
$$\operatorname{tr}(Q(s, x, s, x) + Q(t, y, t, y) - Q(s, x, t, y) - Q(t, y, s, x)) \leqslant C(d(s, t)^{2\alpha} + |x - y|_V^2), \quad \forall s, x, t, y,$$

(iii)
$$|X(s, x)|_V \leqslant C, \quad \forall s, x,$$
$$|X(s, x) - X(t, y)|_V \leqslant C(d(s, t)^{\alpha} + |x - y|_V), \quad \forall s, x, t, y.$$

Then μ and X determine a family of Borel probability measures $\{\nu_t: t \geqslant 0\}$ on $C(S \times M, M)$ satisfying

(a) $\nu_s * \nu_t = \nu_{s+t}$, $\forall s, t \geqslant 0$,

(b) *the ν_t are transition probabilities for a Markov process on $C(S \times M, M)$ with continuous sample paths.*

We illustrate the dependence of the $\{\nu_t\}$ on μ and X as follows. For $\mathbf{s} = (s_1, \ldots, s_r) \in S^r$ and $\mathbf{x} = (x_1, \ldots, x_r) \in M^r$ denote by

$$\rho_{\mathbf{s},\mathbf{x}}: C(S \times M, M) \longrightarrow M^r$$

$$\sigma_{\mathbf{s},\mathbf{x}}: C(\xi) \longrightarrow T_{x_1}M \times \cdots \times T_{x_r}M,$$

the evaluation maps at $(s_1, x_1), \ldots, (s_r, x_r)$. Let $\nu^t_{\mathbf{s},\mathbf{x}}$ be the image of ν_t under $\rho_{\mathbf{s},\mathbf{x}}$, then

(i) the $\nu^t_{\mathbf{s},\mathbf{x}}$ for all $\mathbf{s}$, $\mathbf{x}$ determine ν_t,

(ii) the $\nu^t_{\mathbf{s},\mathbf{x}}$ for fixed $\mathbf{s}$ are the transition probabilities for a Markov process on M^r with continuous sample paths.

THEOREM 2. *The Markov process corresponding $\{\nu^t_{\mathbf{s},\mathbf{x}}: t \geqslant 0, \mathbf{x} \in M^r\}$ has infinitesimal generator $A_{\mathbf{s}}$, where*

$$(A_{\mathbf{s}}g)(\mathbf{x}) = \frac{1}{2}\int(\nabla^2 g)(\mathbf{x})\,(\sigma_{\mathbf{s},\mathbf{x}}(h), \sigma_{\mathbf{s},\mathbf{x}}(h))\,d\mu(h) + (\nabla g)(\mathbf{x})(\sigma_{\mathbf{s},\mathbf{x}}(X)),$$

where ∇ is covariant differentiation with respect to the product Riemannian structure on M^r.

3. **The construction.** For each $\mathbf{s}$, $\mathbf{x}$ and $t > 0$, we construct a measure $\nu^t_{\mathbf{s},\mathbf{x}}$ on M^r as follows. Using the embedding $M \subset V$ and choosing suitable extensions, we construct a Wiener process $\widetilde{W}_t$ in $C(S \times V, V)$ (see Gross [2]) and $\widetilde{X} \in C(S \times V, V)$. Define

$$Y(s, x) = \frac{1}{2}\int_{C(\xi)} h(x)(g(s, x), g(s, x))\,d\mu(g) \in T_x^{\perp}M \quad \text{for } x \in M,$$

and extend to $\widetilde{Y} \in C(S \times V, V)$. Consider the stochastic differential equation in V^r

$$\left.\begin{aligned} d\eta_i(t) &= (\widetilde{X} + \widetilde{Y})(s_i, \eta_i(t))\,dt + dW(t)\,(s_i, \eta_i(t)) \\ \eta_i(0) &= x_i \end{aligned}\right\} i = 1, \ldots, r.$$

The choice of $\widetilde{Y}$ ensures that if $x_i \in M$, then $\eta_i(t) \in M$ for all $t > 0$ with probability one. The conditions (i), (ii) and (iii), plus care in choosing extensions, imply that the equation has a solution for all $t > 0$, that the solution is continuous with probability one and has finite moments of all orders. We define $\nu^t_{\mathbf{s},\mathbf{x}}$ to be the distribution of $(\eta_1(t), \ldots, \eta_r(t)) \in M^r$. The existence of the $\{\nu_t\}$ follows from the Daniell-Kolmogorov construction and an estimate on the moments of solutions of the stochastic differential equation.

4. **Examples.** Suppose S and M are compact Riemannian manifolds and $p > \frac{1}{2}\dim S$, $q > \frac{1}{2}\dim M + 1$. Then $L^2_p(S) \otimes L^2_q(TM) \subset C(\xi)$ is radonifying, and the Wiener measure μ satisfies the conditions of Theorem 1. Take $X = 0$. Each pair (p, q) in the range above yields a different family $\{\nu_t\}$ of measures on $C(S \times M, M)$.

The condition that M be compact may be replaced by completeness together with certain curvature conditions.

Notice that the case $M = \mathbf{R}^n$ yields Gaussian measures. Also S = point and suitable choice of μ gives Brownian motion on M under the sole condition that there exists a closed isometric embedding with $\|h(x)\| \leqslant C(1 + d(x, x_0))$ for some $C > 0$ and $x_0 \in M$.

REFERENCES

1. P. Baxendale, *Gaussian measures on function spaces,* Amer. J. Math. (to appear).

2. L. Gross, *Potential theory on Hilbert space*, J. Functional Analysis 1 (1967), 123–181. MR 37 #3331.

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