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BOUNDARY VALUE PROBLEMS FOR EVEN ORDER NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

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We outline an operator-theoretic method for proving the existence of solutions of boundary value problems where the nonlinearity satisfies a Nagumo condition. In this note we explain the ideas by considering a particular fourth order problem. The method involves (i) converting the boundary value problem to an "alternative problem" [2]; (ii) inducing a splitting of the integral operator in the alternative problem by splitting the original differential operator [3]; and (iii) applying degree theory.

Let S be the Hilbert space $L^2[0, 1]$ with the inner product and norm denoted by $\langle\ ,\ \rangle$ and $\|\cdot\|$ respectively. Let $H^n[0, 1]$ be the Hilbert space of all functions $x(t)$ such that $x(t)$ and its first $(n-1)$ derivatives are absolutely continuous and $x^{(n)}(t) \in L^2[0, 1]$.

THEOREM. *Let $f(t, x, y, z)$ be continuous from $[0, 1] \times R \times R \times R$ into R such that*

(i) $|f(t, x, y, z)| \leqslant a + b|x| + c|y| + Q(|z|)$ *where* Q: $[0, \infty) \to R$ *is a positive continuous nondecreasing function satisfying* $\overline{\lim}_{s\to\infty} Q(s)/s^2 < \infty$; *and*

(ii) *there exists* $R_1 > 0$ *such that* $\|x\| = R_1$, $x \in H^2[0, 1]$ *implies* $\langle x, f(t, x, x', x'')\rangle + \|x''\|^2 \geqslant 0$.

Then the nonlinear boundary value problem

$$x'''' + f(t, x, x', x'') = 0, \qquad x'(0) = x'(1) = x'''(0) = x'''(1) = 0 \tag{1}$$

has at least one solution.

OUTLINE OF PROOF. Let L and T be the linear operators defined by

$$\mathcal{D}(L) = \{x \in H^4[0, 1] : x'(0) = x'(1) = x'''(0) = x'''(1) = 0\}, \quad Lx = x'''',$$

$$\mathcal{D}(T) = \{x \in H^2[0, 1] : x'(0) = x'(1) = 0\}, \quad Tx = x''.$$

Then, if T^* denotes the adjoint of T, it can be seen that $T = T^*$ and $L = TT^*$.

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Let the nonlinear Nemitsky operator N be defined by

$$\mathcal{D}(N) = H^2[0, 1], \quad (Nx)(t) = f(t, x, x', x'').$$

We can rewrite (1) as

$$Lx + Nx = 0. \tag{2}$$

Let S_0 be the null-space of L and $S_1 = S_0^{\perp}$. Let P: $S \longrightarrow S_0$ be the projection operator. By applying the method of "alternative problems," we can transform (2) (cf. [2]) into

$$x + H(I - P)Nx = x^*, \tag{3}$$

$$PNx = 0 \tag{4}$$

where $x^* \in S_0$ and $H = [L|\mathcal{D}(L) \cap S_1]^{-1}$.

The decomposition of L into TT^* induces a decomposition of $H(I - P)$ into J^*J where J^* and J are the appropriate inverses of T^* and T, respectively. If $x_1 \in S_1$ is such that $x = J^*x_1 + x^*$ then (3) and (4) can be written as [3]

$$x_1 + JN(J^*x_1 + x^*) = 0, \tag{5}$$

$$PN(J^*x_1 + x^*) = 0. \tag{6}$$

Solving this system of equations is equivalent to solving

$$(I - R)z = 0 \tag{7}$$

where $z = (x_1, x^*) \in S_1 \oplus S_0$ and R: $S_1 \oplus S_0 \longrightarrow S_1 \oplus S_0$ is defined by

$$Rz = (-JN[J^*x_1 + x^*], x^* - PN[J^*x_1 + x^*]).$$

Now J^* is continuous from S_1 into $H^2[0, 1]$; N maps bounded sets of $H^2[0, 1]$ continuously into bounded subsets of $L^1[0, 1]$, by virtue of the Nagumo condition; and J maps these bounded sets into compact sets in S_1. Hence, R is a compact map of $S_1 \oplus S_0$ into itself. Also,

$$\begin{aligned}\langle (I - R)z, z\rangle_{S_1 \oplus S_0} &= \|x_1\|^2 + \|x^*\|^2 + \langle x_1, JN[J^*x_1 + x^*]\rangle \\ &\quad - \|x^*\|^2 + \langle x^*, PN[J^*x_1 + x^*]\rangle \\ &= \|x_1\|^2 + \langle J^*x_1 + x^*, N[J^*x_1 + x^*]\rangle \\ &\quad - \langle x^*, N[J^*x_1 + x^*] - PN[J^*x_1 + x^*]\rangle \\ &= \|x_1\|^2 + \langle J^*x_1 + x^*, N(J^*x_1 + x^*)\rangle.\end{aligned}$$

At one point we have modified the domains of J^* and P. But these operators are integrals so the modified operators are defined and P is still such that a term vanishes.

Having a solution of (7), we know it is in $L^2[0, 1]$. So in taking it back to (2), P is again a projection. Then we check that the solution is in $C^2[0, 1]$ and hence a solution of (1).

By hypothesis (ii) and a variant of the Borsuk antipodal mapping theorem we conclude that there exists a solution of the equation $(I - R)z = 0$, i.e., there exists a solution of the boundary value problem (1).

REMARKS. By applying the above method to the nonlinear problem

$$x'' = f(t, x, x'), \quad x(0) = x(1), \quad x'(0) = x'(1) \tag{8}$$

we conclude that (8) has at least one periodic solution if

(i) $f(t, x, y)$: $[0, 1] \times R \times R \longrightarrow R$ is continuous and $|f(t, x, y)| \leqslant a + b|x| + Q(|y|)$ where Q is as in the theorem;

(ii) $\langle x, f(t, x, x')\rangle + \|x'\|^2 \geqslant 0$ for all x satisfying $\|x\| = R$.

This result, proved by other methods, may be found in [1] or [4], for example.

We note that for problem (8), $T \neq T^*$. Also, the methods of the theorem may be applied to problems with other boundary conditions.

ADDED IN PROOF. Using that (L_p^*, L_2, L_p) are in so-called normal position we can handle nonlinearities where $Q(s) = O(s^p)$, $p \geqslant 2$.

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