$\Sigma_{n=1}^{N}(j_n(N) - j_n)^2$ goes to zero as N approaches infinity, where $\{j_n\}$ is the solution to equation (2).

These last two theorems are applied in considering the potential problem involving the temperature in a sphere having prescribed temperature in the top half and Newtonian heat loss through the lower half. (In fact, a survey of some seventy papers involving dual orthogonal series shows that these last two theorems are sufficiently general to apply to all of them.)

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CONFORMAL GEOMETRY IN HIGHER DIMENSIONS. I

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Conformally Euclidean manifolds are one type of higher dimensional generalization of Riemann surfaces. They are studied and classified here from that point of view (cf. [2] and [3]).

1. DEFINITION 1.1. *A conformal structure on a manifold M is a covering $\{U_\alpha\}$ together with a metric g_α on U_α such that whenever $U_\alpha \cap U_\beta \neq \emptyset$, g_α and g_β are conformally related on $U_\alpha \cap U_\beta$.*

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An advantage of this notion is that a conformally Euclidean manifold has an *analytic* conformal structure. Moreover

THEOREM 1.2. *A conformally Euclidean manifold admits an analytic Riemann metric compatible with the conformal structure.*

COROLLARY 1.3. *A connected sum of two copies of $S^{n-1} \times S^1$ carries an analytic conformally Euclidean metric which cannot be realized in $\mathbf{R}^{n+1}$ if $n \geqslant 4$ although it carries C^∞ conformally Euclidean metrics which are realizable in $\mathbf{R}^{n+1}$* (cf. [3]).

2. **The model space and the canonical bundle.** The group $\mathfrak{M}_n$ of conformal automorphisms of the standard n-sphere is a Lie group isomorphic to two out of the four components of $O(n+1, 1)$. $(S^n, \mathfrak{M}_n)$ is the model space for n dimensional conformal geometry. Assume $n \geqslant 3$. A basic theorem of Liouville says that a conformal map $U \longrightarrow S^n$, U open nonempty in S^n is the restriction of a unique element of $\mathfrak{M}_n$. It follows that a 1-cocycle arising from a conformal structure on a conformally Euclidean M^n defines an S^n-bundle ξ on M^n with structure group $\mathfrak{M}_n$ and a canonical section σ which in a natural sense is transversal to the base. This bundle is implicit in Kuiper [2] and in much of the work on Kleinian groups (cf. also Gunning [1]). The induced homomorphism ρ: $\pi_1(M) \longrightarrow \mathfrak{M}_n$ is called the *conformal holonomy representation.* If $k = \ker \rho$ and M_k is the corresponding covering space then ξ pulled back to M_k is trivial and the corresponding section σ_k gives rise to the *development* map δ: $M_k \longrightarrow S^n$. We call $\Omega_M = \delta(M_k)$ the *planar model* of M. M is called *Kleinian* (resp. *weakly Kleinian*) if im ρ is discontinuous on Ω_M (resp. discrete in $\mathfrak{M}_n$).

3. **Class and kind.** Write Ω for Ω_M. Let $\partial\Omega$ be its set theoretic boundary. If M is Kleinian and compact $\partial\Omega$ coincides with the closure of the set of fixed points of nonelliptic elements of im ρ. The notation S^p will denote the standard p-sphere if $p \geqslant 0$, a point if $p = -1$, and the empty set if $p = -2$.

DEFINITION 3.1. *M is said to be of class p, $-2 \leqslant p \leqslant n$, if $\partial\Omega$ is contained in some $S^p \subseteq S^n$ but not in any $S^{p-1} \subseteq S^n$. Moreover it is said to be of the first (resp. second) kind if $\partial\Omega = S^p$ (resp. $\partial\Omega \neq S^p$).*

Of course if $p \leqslant 0$ there is no "second kind" and if $p = n$ there is no "first kind." It is seen that $p = -2$ (resp. -1, resp. $n-1$) and first kind corresponds to a spherical (resp. Euclidean, resp. hyperbolic) space form.

THEOREM 3.1. *There exist compact manifolds of all classes and kinds.*

4. **A conformal surgery.** In [3] it was announced that a connected sum of conformally Euclidean manifolds carries a conformally Euclidean structure. We take this opportunity to point out that Professor Milnor pointed out to us that Theorem 1 in [3] is not correct as stated. Since then we have come to realize a close connection of this construction with the "combination theorems" of Klein and Maskit in the theory of Kleinian groups. A general principle may be extracted as follows: Let M^n and N^n be manifolds with homeomorphic boundaries $\partial M \approx^{\varphi} \partial N$. Let $M \#_{\varphi} N$ denote the manifold obtained by glueing M with N along the boundaries by φ. Let ∂M and ∂N have collar neighborhoods $U = \partial M \times [-\epsilon, \epsilon]$, $V = \partial N \times [-\epsilon, \epsilon]$ where ∂M (resp. ∂N) is identified with $\partial M \times \{-\epsilon\}$ (resp. $\partial N \times \{-\epsilon\}$). If there exists a conformal map $H: U \longrightarrow V$ such that $H(x, t) = (f_t(x), -t)$, $x \in \partial M$, $t \in [-\epsilon, \epsilon]$, then identifying U with V via H gives a conformally Euclidean structure on $M \#_{\varphi} N$. This structure *depends* on the map H which reflects on the deformations of a conformal structure on $M \#_{\varphi} N$. An example of such a map H is the inversion in the equatorial $S^{n-1} \subset S^n$. We formulate a general

DEFINITION 4.1. *Let M^{n-1} be a compact smooth submanifold of S^n. An inversion in M^{n-1} is a differentiable involution α: $S^n \longrightarrow S^n$ which leaves M^{n-1} invariant so that the induced map on the normal bundle of M^{n-1} is orientation-reversing. We shall call M^{n-1} a trace of the inversion α.*

THEOREM 4.2. *Given $g = 0, 1, 2, \ldots$: There exist conformal inversions of S^3 whose trace is a compact surface of genus g.*

It is also possible to describe such inversions in higher dimensions. These are possible candidates for the map H above and show the possibility of higher dimensional analogues of the Klein Maskit combination theorems.

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