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# EXTENSION THEOREMS FOR REDUCTIVE GROUP ACTIONS ON COMPACT KAEHLER MANIFOLDS

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Let $G$ be a connected complex reductive Lie group. Noting [3], [7], [8] that $G$ has the structure of a linear algebraic group, let $\bar{G}$ be any projective manifold in which $G$ is Zariski open and which induces the above algebraic structure on $G$. The purpose of the present note is to announce

PROPOSITION I. *Let $G$ be as above and act holomorphically on a compact Kaehler manifold $X$. Assume that the Lie algebra of holomorphic vector fields on $X$ generated by $G$ is annihilated by every holomorphic one form. Let $\Phi$: $Y \longrightarrow X$ be a holomorphic map where $Y$ is a normal reduced analytic space. Consider the equivariant map $\Phi'$: $G \times Y \longrightarrow X$; $\Phi'$ extends meromorphically (in the sense of Remmert) to $\bar{G} \times Y$.*

REMARKS. The condition on vector fields annihilated by one forms is automatically satisfied if (cf. [12]–[14]) $H^1(X, \mathbf{Q}) = 0$, or $G$ is semisimple, or if every generator of the solvable radical of $G$ has a fixed point, or if $G$ is a linear algebraic group acting algebraically on a projective $X$. Taking $Y$ to be a point, one gets the orbits of $G$ to be Zariski open in their closures which are analytic sets. A simple corollary is the classical result that there is only one structure of a linear algebraic group on $G$ (cf. [7]), and in fact any reductive connected subgroup of an algebraic group over $\mathbf{C}$ is an algebraic subgroup.

As a further application of the techniques used, a new proof of an improved form of a fixed point theorem (cf. [12], [13], [14]) of the author is given:

PROPOSITION II. *Let $S$ be a connected solvable Lie group acting holomorphically on a compact Kaehler manifold $X$. The following are equivalent*:

(a) *$S$ has a fixed point on $X$.*

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(b) *$S$ leaves a compact set in a fibre of the Albanese map invariant.*

(c) *$S$ has a fixed point within any compact set $K$ on $X$ that $S$ leaves invariant.*

(d) *The Lie algebra of vector-fields that $S$ generates on $X$ is annihilated by every holomorphic one form on $X$.*

REMARK. The assertion (c) where $K$ is any compact set is new; the method of proof allows one to relax the compactness of $X$ and show if, in addition, $H^1(X, \mathcal{O}_X) = 0$, then (c) is true.

The following is the fundamental observation on which everything rests.

LEMMA. *Let $X$ be a compact Kaehler manifold and $\rho$: $\mathbf{C}^* \longrightarrow \mathrm{Aut}(X)$ a holomorphic $\mathbf{C}^*$ action that has at least one fixed point. Let $A$: $\mathbf{C}^* \longrightarrow X$ be a holomorphic equivariant map onto an orbit: then $A$ extends to a homomorphic equivariant map $\widetilde{A}$ of $\mathbf{C}P^1$ to $X$.*

PROOF. Assume without loss of generality that $A(\mathbf{C}^*)$ is not a point. Let $\mu$ be a Kaehler metric on $X$ and $\omega$ the associated Kaehler form. Assume that $\mu$ has been averaged with respect to the circle subgroup $S^1 \subseteq \mathbf{C}^*$. Let $\chi$ be the holomorphic vector-field on $X$ associated to $\rho$: $\mathbf{C}^* \longrightarrow \mathrm{Aut}(X)$.

Because of equivariance, the Jacobian, $dA$, of $A$, maps some constant multiple of $z(\partial/\partial z)$ onto the restriction of the vector-field $\chi$ to $A(\mathbf{C}^*)$. Without loss of generality this constant is assumed to be one.

Let $A^*\mu = a(r)\, dz \otimes d\bar{z}$ where $a(r)$ is positive and depends only on $r$ due to the $S^1$ averaging of $\mu$. $A^*\omega = (i/2)a(r)\, dz \wedge d\bar{z}$.

$$\mu(\chi, \chi) = \mu\left(dA\left(z\frac{\partial}{\partial z}\right), dA\left(z\frac{\partial}{\partial z}\right)\right) = A^*\mu\left(z\frac{\partial}{\partial z}, z\frac{\partial}{\partial z}\right)$$

$$= a(r)|z|^2 \leqslant M < \infty$$

where $\sup_X \mu(\chi, \chi) = M < \infty$.

Now by Lichnerowicz [5] there exists a $C^\infty$ function $\phi$ on $X$ such that $\bar{\partial}\phi = \omega(\chi)$. Pulling back and, without confusion, letting $\phi$ stand for $A^*\phi = \phi(A(z))$, one has

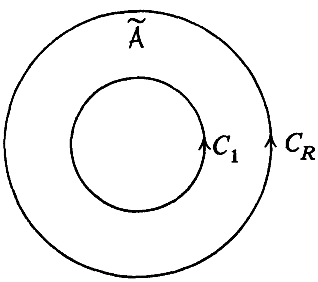
$$\frac{i}{2} z\, a(r)\, d\bar{z} = \frac{\partial\phi}{\partial\bar{z}}\, d\bar{z} \quad \text{or} \quad \frac{i}{2} z\, a(r) = \frac{\partial\phi}{\partial\bar{z}}.$$

Now fix one circle, say the unit circle $C_1 \subset \mathbf{C}^*$ and let $C_R = \{z \in \mathbf{C}^* \mid |z| = R\}$. Assume $R > 1$; $C_1$ and $C_R$ bound an annulus $\widetilde{A}$ with $\partial\widetilde{A} = C_R - C_1$. Now

$$\iint_{\tilde{A}} A^*\mu = \iint_{\tilde{A}} \frac{i}{2} a(r)\, dz \wedge d\bar{z} = -\iint_{\tilde{A}} \frac{\partial \phi}{\partial \bar{z}} \frac{d\bar{z} \wedge dz}{z}$$

$$= -\int_{C_R} \phi \frac{dz}{z} + \int_{C_1} \phi \frac{dz}{z} = \frac{1}{i} \int_0^{2\pi} \phi(Re^{i\theta})\, d\theta - C$$

with $C$ a constant. Now $|\int_0^{2\pi} \phi(Re^{i\theta})\, d\theta| \leqslant M' < \infty$ since $\phi$ is the pullback of a bounded function on $X$.




Therefore $\iint_{\tilde{A}} A^*\mu \leqslant M'' < \infty$ where $M''$ is a positive constant independent of $R$. Thus by Bishop's extension theorem (cf. [1], [2]), $A$ extends holomorphically over $\infty$. An identical argument gives extension at 0. Q.E.D.

Using the above Lemma and the Levi-Griffiths-Shiffman-Siu extension theorem (cf. [2], [9], [10], [11]) repeatedly, one proves the result for SL(2, **C**) and groups of the form $(\mathbf{C}^*)^n$ that have a fixed point on X. Then one proves it for one parameter unipotent subgroup of $G$ by using the above SL(2, **C**) result on an SL(2, **C**) in $G$ containing the subgroup; this can be done by Jacobson-Morosow (cf. [4]). One now proves it for a Borel subgroup of $G$ and uses an argument depending on the fact that one has a locally trivial fibring of $G$ over $G/B$ which is compact.

In the very interesting paper [6] of Lieberman, related matters are discussed.

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