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A RESTRICTION THEOREM FOR THE FOURIER TRANSFORM

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Let f be a Schwartz function on $\mathbf{R}^n$, and let $\hat{f}(\theta)$ denote the restriction of the Fourier transform of f to the unit sphere S^{n-1} in $\mathbf{R}^n$. We prove

THEOREM. *If f is in $L^p(\mathbf{R}^n)$ for some p with $1 \leqslant p < 2(n+1)/(n+3)$, then*

$$\int_{S^{n-1}} |\hat{f}(\theta)|^2 \, d\theta \leqslant c_p \|f\|_p^2.$$

PROOF.

$$\int |\hat{f}(\theta)|^2 \, d\theta = \int f * \tilde{f}(x) \widehat{d\theta}(x) \, dx = \int f(x) \, \widehat{d\theta} * f(x) \, dx \leqslant \|f\|_p \|\widehat{d\theta} * f\|_{p'}$$

for conjugate indices p and p'. Thus it suffices to prove that the operator given by convolution with $\widehat{d\theta}$ is bounded from L^p to $L^{p'}$ for p in the appropriate range. Let $K(x)$ be a radial Schwartz function with $K(x) = 1$ for $|x| \leqslant 100$, and let $T_k(x) = [K(x/2^k) - K(x/2^{k-1})] \; \widehat{d\theta}(x)$. It suffices to show there exists $\epsilon = \epsilon(p) > 0$ such that $\|T_k * f\|_{p'} \leqslant C2^{-\epsilon k}\|f\|_p$. This follows from interpolating the estimates $\|T_k * f\|_\infty \leqslant C2^{-(n-1)k/2}\|f\|_1$ and $\|T_k * f\|_2 \leqslant 2^k\|f\|_2$.

Professor E. M. Stein has extended the range of this result to include $p = 2(n + 1)/(n + 3)$. His proof uses complex interpolation of the operators given by convolution with the functions $B_\sigma(x) = J_\sigma(2\pi|x|)/|x|^\sigma$. Then $\widehat{d\theta}(x) = B_{(n-2)/2}(x)$.

A great deal was previously known about such restriction theorems. E. M. Stein originally established the theorem for $1 \leqslant p < 4n/(3n + 1)$. For $n = 2$, this was extended by Fefferman and Stein [2] to the range $1 \leqslant p < 6/5$. P. Sjolin (see [1]) proved the theorem for $n = 3$ and $1 \leqslant p \leqslant 4/3$. Finally, A. Zygmund [3] determined for two dimensions all p and q such that the Fourier transform of an L^p function restricts to $L^q(S^1)$. Since a

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very simple homogeneity argument shows that the theorem fails for $p > 2(n + 1)/(n + 3)$, the present result, together with the result of Stein, represent the optimal estimate of this sort.

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