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LINEAR APPROXIMATION BY EXPONENTIAL SUMS ON FINITE INTERVALS

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Let $\Lambda = \{\lambda_k\}_{k=1}^{\infty}$ be a sequence of distinct nonnegative real numbers. It is well known that the exponential sums

$$(1) \qquad e_s(x) = \sum_{k=1}^{s} a_k e^{\lambda_k t}, \qquad a_k \in R, \; s = 1, 2, \cdots,$$

are dense in $C[A, B]$, $-\infty < A < B < +\infty$, if and only if Müntz' condition $\Sigma_{\lambda_k \neq 0} 1/\lambda_k = +\infty$ holds. In this note Jackson-type results on the rate of convergence of the exponential sums (1) are given. Substituting

$$(2) \qquad x = e^{t-B}, \qquad t \in [A, B], \; x \in [a, 1],$$

where $a = e^{A-B}$, we are led to the problem where the functions $f \in C[a, 1]$, $0 < a < 1$, are to be approximated on $[a, 1]$ by the Λ-polynomials

$$(3) \qquad p_s(x) = \sum_{k=1}^{s} b_k x^{\lambda_k}, \qquad b_k \in R, \; s = 1, 2, \cdots.$$

Recently, many optimal or almost optimal Jackson-Müntz theorems on the approximation properties of the Λ-polynomials (3) for the interval [0, 1] have been published (cf. J. Bak and D. J. Newman [1] and M. v. Golitschek [2]). Considering intervals $[a, 1]$, $a > 0$, one would expect that the Λ-polynomials have even better approximation properties than on [0, 1], as the "singular" point $x = 0$ might have less influence. Theorems 1 and 2 prove this conjecture.

THEOREM 1. *Let* $0 < a < 1, M > 0$. *If* Λ *satisfies*

$$(4) \qquad 0 \leqslant \lambda_k \leqslant Mk \quad \text{for all } k = 1, 2, \cdots,$$

then for each function $f \in C^r[a, 1]$, $r \geqslant 0$, *and each integer* $s \geqslant r + 1$ *there exists a* Λ-*polynomial* p_s *such that for all* $a \leqslant x \leqslant 1$

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(5) $$|f(x) - p_s(x)| \leqslant K_r s^{-r} \omega(f^{(r)}; 1/s) + O(\rho^s),$$

where ω denotes the modulus of continuity; $K_r > 0$ depends on a, M, and r; and ρ $(0 < \rho < 1)$ depends only on a and M.

Consequently, if the exponents Λ satisfy (4), the Λ-polynomials behave asymptotically as well as the ordinary algebraic polynomials. As the sth width $d_s(\Lambda_{r\omega})$ of the class $\Lambda_{r\omega}(M_0, \cdots, M_{r+1};\ [a, 1])$ of functions in $C[a, 1]$ is

$$d_s(\Lambda_{r\omega}) \approx s^{-r}\omega(1/s)$$

(cf. G. G. Lorentz [3, Chapters 3.7 and 9.2]), the Λ-polynomials of Theorem 1 approximate asymptotically optimally in this special sense.

EXAMPLE. The exponents Λ with $\lim_{k\to\infty} \lambda_k = \lambda \geqslant 0$ satisfy condition (4). For the corresponding problem in $[0, 1]$ we could only prove (cf. M. v. Golitschek [2, p. 95]) that there exist Λ-polynomials p_s for which

$$|f(x) - p_s(x)| = O(\sqrt{s}^{-r}\omega(f^{(r)}; 1/\sqrt{s})), \qquad s \to \infty.$$

THEOREM 2. *Let $0 < a < 1, M > 0, \epsilon > 0$. Let Λ satisfy*

(6) $$\lambda_k \geqslant Mk \quad \textit{for all} \ \ k = 1, 2, \cdots.$$

For each $s \geqslant s_0$ (s_0 sufficiently large) let $\psi(s)$ be defined as the largest positive integer for which

(7) $$\sum_{\psi \leqslant k \leqslant s} \frac{1}{\lambda_k} \geqslant -(1+\epsilon)\log\sqrt{a}.$$

Then for each $f \in C^r[a, 1]$ and each $s \geqslant s_0$ there exists a Λ-polynomial p_s such that for all $a \leqslant x \leqslant 1$

(8) $$|f(x) - p_s(x)| \leqslant K_r \psi(s)^{-r}\omega(f^{(r)}; 1/\psi(s)) + O(\rho^{\psi(s)}),$$

where K_r depends on a, r, M, and ϵ; and ρ $(0 < \rho < 1)$ depends on a, M, and ϵ.

EXAMPLE. Let $\lambda_k = k\log k$, $k = 1, 2, \cdots, M = 1, \epsilon > 0$. From (7) we obtain

$$\psi(s) \approx s^{\sqrt{a}^{1+\epsilon}}.$$

In [1] and [2] it was proved that in [0, 1] the corresponding "rate of convergence" is only

$$\varphi(s) = \exp\left(-2\sum_{k=1}^{s} \frac{1}{k \log k}\right) \approx (\log s)^{-2}.$$

The above theorems are proved by the same method used by the author in his earlier paper [2] for Jackson-Müntz theorems on the interval $[0, 1]$: First the function f is approximated by ordinary algebraic polynomials P_n and then each monomial x^q $(q = 0, 1, \cdots, n)$ of P_n is approximated by appropriate Λ-polynomials. The full details and further results will be published later.

By the substitution $t = B + \log x$ we obtain from Theorems 1 and 2 immediately the corresponding approximation theorem for the exponential sums (1).

THEOREM 3. *Let $F \in C^r[A, B]$, $-\infty < A < B < +\infty$, $r \geq 0$. Let the best approximation of F be defined by*

$$E_s^*(F; \Lambda) = \inf_{a_k} \max_{A \leq t \leq B} \left| F(t) - \sum_{k=1}^{s} a_k e^{\lambda_k t} \right|. \tag{9}$$

If Λ satisfies (4), *then*

$$E_s^*(F; \Lambda) = O(s^{-r}\omega(F^{(r)}; 1/s)) \quad \textit{for} \quad s \to \infty. \tag{10}$$

If Λ satisfies (6), *then for each $\epsilon > 0$*

$$E_s^*(F; \Lambda) = O(\psi(s)^{-r}\omega(F^{(r)}; 1/\psi(s))) \quad \textit{for} \quad s \to \infty, \tag{11}$$

where $\psi(s)$ is defined by (7) *with* $\log \sqrt{a} = (A - B)/2$.

REMARK. The same results are also valid in the L_p norms, $1 \leq p < \infty$, if the function f (or F) has an $(r - 1)$st absolutely continuous derivative in $[a, 1]$ (or $[A, B]$) and $f^{(r)} \in L_p(a, 1)$ (or $F^{(r)} \in L_p(A, B)$) and if ω denotes the integral modulus of continuity in L_p.

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