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STEENROD HOMOLOGY AND OPERATOR ALGEBRAS

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The recent work of Larry Brown, R. G. Douglas, and Peter Fillmore (referred to as BDF) [2], [3], and [4] on operator algebras has created a new bridge between functional analysis and algebraic topology. This note and a subsequent paper [5] constitute an effort to make that bridge more concrete.

We first briefly describe the BDF framework. This requires the following C^*-algebras: $C(X)$, the continuous complex-valued functions on a compact metric space X; L, the bounded operators on an infinite dimensional separable Hilbert space; $K \subset L$, the compact operators; and L/K, the Calkin algebra. (Let π: $L \longrightarrow L/K$ be the projection.) An *extension* is a short exact sequence of C^*-algebras and C^*-algebra morphisms of the form $0 \longrightarrow K \longrightarrow E \longrightarrow C(X) \longrightarrow 0$ where E is a C^*-algebra containing K and I (the identity operator) and contained in L. Unitary equivalence classes of extensions form an abelian group, denoted $\mathrm{Ext}(X)$.

$\mathrm{Ext}(X)$ was invented by BDF in order to study essentially normal operators, that is, operators $T \in L$ with πT normal. Let E_T denote the C^*-algebra generated by I, T, and K, and let $X = \sigma(\pi T)$, the spectrum of πT. Then the exact sequence $0 \longrightarrow K \longrightarrow E_T \longrightarrow C(X) \longrightarrow 0$ represents an element of $\mathrm{Ext}(X)$. This element is zero if and only if T is a compact perturbation of a normal operator. For $X \subset \mathbf{C}$, BDF prove that

$$\mathrm{Ext}(X) \simeq \widetilde{H}^0(\mathbf{C} - X). \tag{1}$$

This isomorphism assigns to E_T a sequence of integers corresponding to the Fredholm index of $T - \lambda I$ on the various bounded components of $\mathbf{C} - X$.

The isomorphism (1) was subsequently generalized [3]. Let $E_{2n+1}(X) = \mathrm{Ext}(X)$ and $E_{2n}(X) = \mathrm{Ext}(SX)$, where SX is the suspension of X. Then BDF show that E_* satisfies (on compact metric pairs) all of the Eilenberg-Steenrod

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axioms for an homology theory, except the dimension axiom. For finite CW complexes, $E_*(X) = \widetilde{K}_*(X)$, where $\widetilde{K}_*$ is the reduced homology theory corresponding to complex K-theory [1] and [10].

The homology theory E_* satisfies two additional axioms:

(RH) Let f: $(X, A) \longrightarrow (Y, B)$ be a relative homeomorphism (i.e., $f|X - A$ is a homeomorphism onto $Y - B$). Then f_*: $E_*(X, A) \longrightarrow E_*(Y, B)$ is an isomorphism.

(Wedge) Let $\bigvee_j X_j$ be the *strong* wedge of a sequence of pointed compact metric spaces. Then $E_*(\bigvee_j X_j) = \Pi_j E_*(X_j)$. (The strong wedge of a family of pointed spaces is the subspace of the product consisting of all points with at most one coordinate not a basepoint.)

In 1941, Steenrod introduced a homology theory on compact metric spaces via "regular cycles" [9] and [8]. This theory, which we denote by ${}^sH_*$, satisfies all seven of the usual axioms as well as **(RH)** and **(Wedge)**. Steenrod showed that one may obtain ${}^sH_n(X)$ as follows. Write $X = \varprojlim X_j$, where $\{X_j, p_j\}$ is an inverse system of finite simplicial complexes obtained as the nerves of open covers which successively refine each other and whose mesh goes to zero. Assume also that X_0 = point. Let FX be the infinite mapping cylinder—that is, $FX = (\bigcup_j X_j \times [j, j + 1])/\sim$, where $\sim$ is the equivalence relation corresponding to pasting the cylinders $X_j \times [j, j + 1]$ together at their ends via the maps $\{p_j\}$. Then FX admits the structure of a countable, locally finite CW-complex. Steenrod proved that ${}^sH_n(X)$ is isomorphic to the $(n + 1)$st homology group of FX based on infinite chains. We thus obtain a useful characterization of the groups ${}^sH_*(X)$.

Steenrod [9] and Milnor [7] also proved that ${}^sH_n(X)$ is related to the more common Čech homology group $\check{H}_n(X) = \varprojlim_j H_n(X_j)$ by a split exact sequence

$$0 \longrightarrow \varprojlim{}^1 H_{n+1}(X_j) \longrightarrow {}^sH_n(X) \longrightarrow \check{H}_n(X) \longrightarrow 0. \tag{2}$$

Milnor also showed that ${}^sH_*$ is the dual theory to Čech cohomology theory on compact metric spaces. Since E_* bears the same relationship to cohomology K-theory on compact metric spaces, we were led to make precise the relation between E_* and ${}^sH_*$.

An important tool is the spectral sequence provided by the following theorem.

THEOREM 1. *Let X be compact metric of dimension $d < \infty$. Then there is a spectral sequence $\{E^r_{p,q}\}$ which converges to $E_*(X)$, is natural in X,* *has* $E^{d+1} = E^{\infty}$ *and* $E^2_{p,q} = {}^s\widetilde{H}_p(X; \mathcal{E}_q(point))$.

For finite CW-complexes this spectral sequence is equivalent to the Atiyah-Hirzebruch spectral sequence.

If $X \subset R^3$ then $\mathcal{E}_*(X)$ is determined by Steenrod homology. Precisely, $\mathcal{E}\mathrm{xt}(X) = {}^s\widetilde{H}_1(X)$ and there is an exact sequence $0 \longrightarrow {}^s\widetilde{H}_0(X) \longrightarrow \mathcal{E}_0(X) \longrightarrow {}^s\widetilde{H}_2(X) \longrightarrow 0$. This is useful in studying the following question. Let A_1 and A_2 be essentially normal operators such that πA_1 and πA_2 commute. When do there exist compact perturbations $A_j = B_j + K_j$, $j = 1, 2$, with B_1 and B_2 commuting normals? If A_2 is selfadjoint then the obstruction to perturbation is an element of $\mathcal{E}\mathrm{xt}(X) = {}^s\widetilde{H}_1(X)$, where $X =$ joint $\sigma(\pi A_1, \pi A_2) \subset R^3$. So, for example, if ${}^s\widetilde{H}_1(X) = 0$ then the B_j exist. If A_2 is just normal then $X \subset R^4$ and the obstruction group $\mathcal{E}\mathrm{xt}(X)$ is an extension of ${}^s\widetilde{H}_1(X)$ by a certain subgroup of ${}^s\widetilde{H}_3(X)$. The applicability of higher dimensional computations to operator theory was first observed by BDF [4, p. 119].

In analogy to K-theory there is a Chern character useful in comparing $\mathcal{E}_*$ with homology. This yields ch $\otimes Q$: $\mathcal{E}_*(X) \otimes Q \longrightarrow {}^s\widetilde{H}_*(X; Q)$ which is not always an isomorphism, in contrast to the cohomology K-theory situation.

THEOREM 2. *The following are equivalent*:

(a) *The differentials in* $\{E^r_{p,q}\}$ *are torsion-valued and* ch $\otimes Q$ *is an isomorphism.*

(b) $\hom(\check{H}^*(X), Q/Z) \otimes Q = 0$.

Finally, an analog of (2) holds for $\mathcal{E}_*$. If X is the inverse limit of finite CW-complexes X_j, then there is a split exact sequence

$$0 \longrightarrow \varprojlim{}^1 \widetilde{K}_0(X_j) \longrightarrow \mathcal{E}\mathrm{xt}(X) \longrightarrow \varprojlim \widetilde{K}_1(X_j) \longrightarrow 0$$

and thus $\mathcal{E}\mathrm{xt}(X)$ is completely determined by K-theory on finite complexes. Also, if X and Y have the same shape [6] then $\mathcal{E}_*(X) \simeq \mathcal{E}_*(Y)$.

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