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A RADIAL MULTIPLIER AND A RELATED KAKEYA MAXIMAL FUNCTION

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In this paper we state some results for a maximal function and a Fourier multiplier that are connected with the Bochner-Riesz spherical summation of multiple Fourier series (see Fefferman [3], [5]). Our purpose will be to get sharp estimates for the norm of these operators in dimension two. Proofs will appear elsewhere [2].

Let $N \geqslant 1$ be a real number. By a rectangle of eccentricity N we mean a rectangle R such that

$$\frac{\text{Length of the bigger side of } R}{\text{Length of the smaller side of } R} = N.$$

We will define the direction of R as the direction of its bigger side.

Given a locally integrable function f we consider the maximal function

$$Mf(x) = \operatorname*{Sup}_{x \in R} \frac{1}{|R|} \int_R |f(y)|\, dy,$$

where the "Sup" is taken over rectangles of eccentricity N, but arbitrary direction.

THEOREM 1. *The sublinear operator M is bounded in $L^2(R^2)$ and there exists a constant C, independent of N, such that*

$$\|Mf\|_2 \leqslant C(\log 3N)^2 \|f\|_2.$$

Suppose that m_0 is a smooth function on R with support on $(-1, 1)$ and let $m(r) = m_0(\delta^{-1}(r-1))$, where $\delta > 0$ is a small number.

Consider the Fourier multiplier defined by

$$\widehat{Tf}(\xi) = m(|\xi|)\hat{f}(\xi), \quad f \in C_0^\infty(R^2).$$

THEOREM 2. *There exists a constant C, independent of δ, such that*

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$$\|Tf\|_4 \leqslant C|\log \delta|^{5/4}\|f\|_4, \quad \forall f \in C_0^\infty(R^2).$$

Our proofs of Theorems 1 and 2 are made in the spirit of Cotlar's lemma. In particular the support of the kernel of T can be decomposed into a family of rectangles of eccentricity $\delta^{-1/2}$ and the convolution operators, obtained by restricting the kernel to these rectangles, are "almost orthogonal".

Theorem 2 can be applied to get Bochner-Riesz summability below the critical index, in dimension two.

COROLLARY 3 (CARLESON-SJOLIN-FEFFERMAN-HÖRMANDER). *The operator T_λ defined by $\widehat{T_\lambda f}(\xi) = m_\lambda(\xi)\hat{f}(\xi)$, where $m_\lambda(\xi) = (1 - |\xi|^2)^\lambda$ if $|\xi| \leqslant 1$ and $m_\lambda(\xi) = 0$ otherwise, is bounded in $L^p(R^2)$ if*

$$\frac{4}{3+2\lambda} < p < \frac{4}{1-2\lambda}, \quad \frac{1}{2} > \lambda > 0.$$

To see this we define a partition of unity on $(0, 1)$ as follows: For every n, h_n is a smooth function with support on $(1 - 2^{-n+1}, 1 - 2^{-n-1})$ such that $|D^p h_n(r)| \leqslant A_p 2^{np}$ (with A_p independent of n) and $\Sigma\, h_n(r) = 1$ on $(0, 1)$. Then $m_\lambda(\xi) = \Sigma\, m_\lambda(\xi)h_n(|\xi|)$. If we apply Theorem 2 to the operator T_λ^n defined by the multiplier $m_\lambda(\xi)h_n(|\xi|)$ we get that $\|T_\lambda^n f\|_4 \leqslant C2^{-n\lambda}n^{5/4}\|f\|_4$. And then, Corollary 3 can be deduced from this estimate by standard arguments of interpolation, duality and adding a geometric series.

REMARK. Theorem 2 can be used to prove a sharper version of Corollary 3 i.e., suppose that m is a smooth function on $(0, 1)$ such that it behaves like

$$\left(\log \frac{1}{1-|x|}\right)^{-t} \quad \text{near } |x| = 1.$$

Then m is a multiplier for $L^p(R^2)$, $4/3 \leqslant p \leqslant 4$ provided that $t > 9/4$.

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REFERENCES

1. L. Carleson and P. Sjolin, *Oscillatory integrals and a multiplier problem for the disc*, Studia Math. 44 (1972), 287–299.

2. A. Cordoba, *The Kakeya maximal function and the spherical summation multipliers* (to appear).

3. C. Fefferman, *Inequalities for strongly singular convolution operators*, Acta Math. 124 (1970), 9–36. MR 41 #2468.

4. ———, *The multiplier problem for the ball*, Ann. of Math. (2) 94 (1971), 330–336. MR 45 #5661.

5. C. Fefferman, *A note on spherical summation multipliers*, Israel J. Math **15** (1973), 44–52. MR **47** #9160.

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