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PREHOMOGENEOUS VECTOR SPACE DEFINED BY A SEMISIMPLE ALGEBRAIC GROUP

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1. **Introduction.** Let G be an affine algebraic group defined over $\boldsymbol{C}$, and π a rational representation of G in a finite-dimensional vector space V. Following the terminology of [**6**] and [**7**], we say that V is a prehomogeneous vector space if π has a (unique) open orbit. In [**7**] F. J. Servedio gave some characterization of the stabilizer G_0 defined by an irreducible prehomogeneous vector space (G, V). Namely, he has proven that G_0 is reductive iff (G', V) is not a prehomogeneous vector space, where G' is the (semisimple) commutator subgroup of G.

It follows from this characterization that if (G, V) is an irreducible prehomogeneous vector space with G semisimple, then the corresponding stabilizer is not reductive. In this paper we would like to determine the stabilizer more affirmatively in the case G is semisimple.

2. **Statements of the main results.** The following theorem will give a proper determination of the stabilizer mentioned above.

THEOREM 1. *Let (G, V) be a prehomogeneous vector space with G semisimple, let G_0 be the corresponding stabilizer. Then the radical of G_0 is a nontrivial unipotent subgroup of G_0.*

This result turns out to be the key for the determination of infinite-dimensional square-integrable representations of certain classes of groups called the U-groups [**1**].

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[1] The Society was unable to locate this author; corrections were done by Professor Hyman Bass.

On the other hand it should be noted that there is no restriction on the irreducibility of (G, V). The main reason is that in the proof of Theorem 1, which uses an induction on dim V, we are led to consider the case in which (G, V) is not necessarily irreducible. In fact we are also forced to treat the case in which G is not semisimple and to replace the representation determining (G, V) by a 1-cocycle. More exactly the groups under consideration are the U-groups defined as follows.

DEFINITION. Let G be an affine algebraic group defined over C. Then G is said to be a *U-group* if the radical of G is unipotent.

According to [5], such a group is necessarily the semidirect product of a connected unipotent group and a semisimple group.

We are now ready to state the following result which covers one half of Theorem 1.

THEOREM 2. *Let G be a U-group, π a rational representation of G in a vector space V, and λ a rational V-valued* 1*-cocycle of G with respect to π. Assume that $\{\lambda(g); g \in G\}$ is open in V. Then $G_0=\{g \in G; \lambda(g)=0\}$ is a U-subgroup of G.*

Now to prove the other half of Theorem 1 it is necessary to extend the stated result in [7] to nonirreducible spaces. This is done with an induction argument similar to the proof of Theorem 2 and thus requires extending the consideration to U-groups. We have

THEOREM 3. *Let G, π, λ, G_0 be as in Theorem 2. Then G_0 is semisimple iff λ is an isomorphism of the affine varieties* Rad G *and V. Moreover, in that case, G_0 is equal to a maximal semisimple subgroup of G.*

3. **Sketch of the proof of Theorem 2.** Since the problem is a purely local one, we can assume G to be connected and simply connected and effectively use the representation theory of semisimple Lie algebras in [3], [4].

By considering subrepresentations and quotient representations if necessary, we can assume that π is irreducible. In this case, since $\lambda(N)$, N being the radical of G, is a π-invariant subspace, we must have $\lambda(N)=V$ or $\lambda(N)=0$.

In the first case it can be proved that G_0 contains a maximal semisimple subgroup of G and is therefore a U-group. On the other hand, by using Dynkin's classification of maximal subalgebras of semisimple Lie algebras in [3], we can reduce the second case to that of a simple group.

In fact let G be the direct product of G_1 and G_2, and π the tensor product of two irreducible representations π_1 and π_2 of degrees n_1 and n_2 of G_1 and G_2, respectively, such that $n_1 \geqq n_2$. Let $\mathfrak{g}_1$, $\mathfrak{g}_2$, $\mathfrak{g}$ be the Lie algebras of

G_1, G_2, G, respectively, and $d\pi_1$, $d\pi_2$, $d\pi$ be the corresponding Lie algebra representations.

Let us write every element of $\mathfrak{sl}(n_1, C)$ as a 2×2-block matrix (g_{ij}), where g_{11} and g_{22} are square matrices of orders n_2 and n_1-n_2, respectively, such that $\operatorname{Tr} g_{11}+\operatorname{Tr} g_{22}=0$. Put

$$\mathfrak{p}_{n_2,n_1-n_2} = \{(g_{ij}) \in \mathfrak{sl}(n_1, C); g_{21} = 0\},$$

and

$$\mathfrak{p}' = \{(g_{ij}) \in \mathfrak{p}_{n_2,n_1-n_2}; g_{11} \in d\pi_2(\mathfrak{g}_2)\}.$$

By identifying the representation space of π with the set of those block matrices $(g_{ij}) \in \mathfrak{gl}(n_1, C)$ such that g_{12} and g_{22} are 0 matrices. We see that π has an open orbit iff $\mathfrak{sl}(n_1, C)=d\pi_1(\mathfrak{g})+\mathfrak{p}'$ and the corresponding stabilizer is a U-group iff $d\pi_1(\mathfrak{g})\cap\mathfrak{p}'$ is a U-algebra. Thus Theorem 2 is equivalent to

THEOREM 2′. *Let* $\mathfrak{g}=\mathfrak{sl}(n, C)$ *and* $\mathfrak{g}'$, $\mathfrak{g}''$ *be two U-subalgebras of* $\mathfrak{g}$ *such that* $\mathfrak{g}=\mathfrak{g}'+\mathfrak{g}''$ *(as the sum of vector spaces), then* $\mathfrak{g}'\cap\mathfrak{g}''$ *is a U-algebra.*

Here we have naturally used the term U-algebra to indicate the Lie algebra of a U-group. $\mathfrak{g}'$ and $\mathfrak{g}''$ are then the Lie algebras of U-subgroups of $SL(n, C)$ [2].

Now in the proof of Theorem 2′ it can be shown that at least one of the Lie algebras $\mathfrak{g}'$ or $\mathfrak{g}''$ is reducible. Moreover, by an induction on rank $\mathfrak{g}$ (see the remark), it is sufficient to consider the case in which $\mathfrak{g}''$ is a maximal reducible U-algebra of the form $\mathfrak{p}_{n_1,n_2}$, $n_1+n_2=n$ and $\mathfrak{g}'$ is either:

(1) maximal irreducible nonsimple of the form $\mathfrak{sl}(p, C)\otimes\mathfrak{sl}(q, C)$, $pq=n$, or

(2) irreducible simple.

By the same reasoning as that preceding Theorem 2′, we see that the study of case (1) is equivalent to that of the irreducible prehomogeneous vector space defined by $\pi=\pi_1\otimes\pi_2\otimes\pi_3$ where π_1, π_2, π_3 are the canonical representations (or their contragredients) of $SL(n, C)$, $SL(p, C)$, $SL(q, C)$, respectively. For reasons of symmetry and by induction it is sufficient to consider the case $n\geqq p\geqq q$ and $n\geqq \frac{1}{2}pq$. Keeping in mind the condition: dim of stabilizer$=$dim $G-\deg \pi$, we find the necessary condition for the existence of the open orbit: $n=p=q=2$. However in this case $\pi_1\otimes\pi_2$ is an orthogonal representation and we can apply the following

LEMMA. *Let G be the direct product of the semisimple groups G_1 and G_2 and $\pi=\pi_1\otimes\pi_2$ where π_1 is an orthogonal representation of G_1 and π_2 is a representation of G_2 such that* $\deg \pi_1\geqq\deg \pi_2$. *Then π has no open orbit.*

Note that the above lemma also takes care of part of Case (2) in which $\mathfrak{g}'$ is an irreducible subalgebra of the orthogonal algebra $\mathfrak{so}(n, C)$. Since most of irreducible simple subalgebras of $\mathfrak{sl}(n, C)$ of low dimensions are subalgebras of the orthogonal algebra this leaves only a small number of cases for the direct computation. The following table gives the list of nonorthogonal simple irreducible subalgebras $\mathfrak{g}'$ such that $\mathfrak{g}'+\mathfrak{g}''=\mathfrak{sl}(n, C)$, $\mathfrak{g}''=\mathfrak{p}_{n_1,n_2}$

type of $\mathfrak{g}'$	rep. defining $\mathfrak{g}'$	type of $\mathfrak{g}''$	Rad $(\mathfrak{g}'\cap\mathfrak{g}'')$	max. s. s. sub-alg. of $\mathfrak{g}'\cap\mathfrak{g}''$
A_{2k} $n-k(2k-1)$	$\circ-\overset{1}{\circ}-\circ\cdots\circ$	$\mathfrak{p}_{1,n-1}$	unipot. abel. of dim $2k$	C_k
A_{2k} $n-k(2k-1)$	$\circ-\overset{1}{\circ}-\circ\cdots\circ$	$\mathfrak{p}_{2,n-2}$	unipot. of dim $2k$	0
$C_{n/2}$	$\overset{1}{\circ}-\circ\cdots\circ$	$\mathfrak{p}_{2k-1,n-2k-1}$	unipot. abel. of dim $n-1$	$C_k \times C_{n/2-k-1}$
D_5	$\circ-\circ-\circ\begin{smallmatrix}\overset{1}{\circ}\\ \circ\end{smallmatrix}$	$\mathfrak{p}_{1,15}$	unipot. abel. of dim 8	B_3

4. Remark. (1) In the proof of the equivalence of Theorems 2 and 2′, it is necessary to extend Theorem 2′ to the case in which $\mathfrak{g}$ is a U-subalgebra whose maximal semisimple subalgebra is the direct sum of algebras of type A_n. It would be very interesting if we could prove Theorem 2′ for any simple algebra. If this could be done then Theorem 2′ remains valid for any U-algebra.

(2) Despite the proofs of Theorems 2 and 2′ coming very close to characterizing the irreducible prehomogeneous vector spaces defined by an arbitrary semisimple group, we have not yet succeeded in achieving the complete classification of those spaces.

References

1. N. H. Anh, *Algebraic groups with square integrable representations*, Bull. Amer. Math. Soc. (to appear).

2. A. Borel, *Linear algebraic groups*, Benjamin, New York, 1969. MR **40** #4273.

3. E. B. Dynkin, *Semisimple subalgebras of semisimple Lie algebras*, Mat. Sb. **30** (72) (1952), 349–462; English transl., Amer. Math. Soc. Transl. (2) **6** (1957), 111–244. MR **13**, 904.

4. A. I. Mal′cev, *On semisimple subgroups of Lie groups*, Izv. Akad. Nauk SSSR Ser. Mat. **8** (1944), 143–174; English transl., Amer. Math. Soc. Transl. (1) **9** (1962), 172–213. MR **6**, 146.

5. G. D. Mostow, *Fully reducible subgroups of algebraic groups*, Amer. J. Math. **78** (1956), 200–221. MR **19**, 1181.

6. Mikio Sato, *The theory of prehomogeneous vector spaces*, Sûgaku **15** (1970), 85–157.

7. F. J. Servedio, *Prehomogeneous vector spaces and varieties*, Trans. Amer. Math. Soc. **176** (1973), 421–444.

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