

NEW INFINITE FAMILIES IN THE
 STABLE HOMOTOPY OF SPHERES

BY LARRY SMITH¹

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Let p be an odd prime and denote by π_*^s the stable homotopy ring of spheres. There are known two infinite families of elements

$$\alpha_t \in \pi_*^{2t(p-1)-1}, \quad \beta_t \in \pi_*^{2t(p^2-1)-2p}$$

introduced in [4] and [2I]; the latter sequence only when $p > 3$. In [5] Toda also constructed elements

$$\epsilon_i \in \pi_*^{2(p-1)(p^2+i)-2} \quad i = 1, \dots, p-1,$$

and by a seemingly numerical accident in Toda's tables (at least up to a non-zero multiple) $\epsilon_{p-1} = \beta_p$. Thus the element ϵ_{p-1} has a bordism theoretic interpretation. In this note we announce the results of our investigation of the remaining elements $\epsilon_1, \dots, \epsilon_{p-2}$. To describe our results we recall some notations and results of [2].

Let $\Omega_*^U(\)$ denote the complex bordism homology theory [3]. Recall that the coefficient ring is

$$\Omega_*^U \simeq \mathbf{Z}[x_2, x_4, \dots]; \quad \deg x_{2i} = 2i$$

an infinite polynomial ring. Polynomial generators of dimension $2p^i - 2$ play a special role in the theory, are called *Milnor manifolds*, and are usually denoted by \mathbf{V}^{2p^i-2} .

Following the stable notations of [2] let $V(0) = S^0 \cup_p e^1$, so that $\tilde{\Omega}_*^U(V(0)) \simeq \tilde{\Omega}_*^U(p)$. There is the map [2I; 1.5] $\bar{\varphi}: S^{2p-2}V(0) \rightarrow V(0)$ such that $\bar{\varphi}_*: \tilde{\Omega}_*^U(S^{2p-2}V(0)) \rightarrow \tilde{\Omega}_*^U(V(0))$ is given by

$$\bar{\varphi}_*(\Sigma^{2p-2}\gamma_0) = [\mathbf{CP}(p-1)]\gamma_0,$$

where $\gamma_0 \in \tilde{\Omega}_0^U(V(0))$ is a generator. There is the iteration $\bar{\varphi}^s: S^{2s(p-1)}V(0) \rightarrow V(0)$ whose mapping cone we denote by $V^{(s)}(1)$. (N.B. when $s = 1$

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$V^{(1)}(1)$ is just the space $V(1)$ of [2I].

Our first result describes the elements $\epsilon_1, \dots, \epsilon_{p-1}$ in bordism theoretic terms.

THEOREM 1. *Let $p > 3$ be a prime. Then there exists a stable map*

$$\hat{\epsilon}_r: S^{2p(p^2-1)} \rightarrow V^{(p-r)}(1)$$

such that

$$\begin{array}{ccc} S^{2p(p^2-1)} & \xrightarrow{\hat{\epsilon}_r} & V^{(p-r)}(1) \\ \epsilon_r \searrow & & \downarrow c \\ & & S^{2(p-r)(p-1)+2} \end{array}$$

is commutative. (N.B. The cell structure of $V^{(p-r)}(1)$ is $S^0 \cup_p e^1 \cup e^{2(p-r)(p-1)+1} \cup_p e^{2(p-r)(p-1)+2}$, and c is collapse onto the top cell.)

Moreover

$$[S^{2p(p^2-1)}, \hat{\epsilon}_r] = ([V^{2p^2-2}]^p + [M])\gamma_0 \in \tilde{\Omega}_*^U(V^{(p-r)}(1)),$$

where $[M] \in ([\mathbb{C}P(p-1)]^{p-r-1}) \subset \Omega_*^U$. \square

Note that for $r = p - 1$, recalling $\epsilon_{p-1} = \beta_p$, we see that Theorem 1 is contained in [2I, §5]. This serves as the starting point for the proof which is completed by very careful use of the calculations of Toda.

The next result advances us to the second stage of the program of [2I] for producing infinite families in π_*^s .

THEOREM 2. *Let $p > 3$ be a prime. Then there is an extension of $\hat{\epsilon}_r$ to $V^{(p-r)}(1)$, that is, there exists a stable map such that the diagram*

$$\begin{array}{ccc} S^{2p(p^2-1)} & V^{(p-r)}(1) & \xrightarrow{\Delta_r} & V^{(p-r)}(1) \\ \uparrow & \nearrow \hat{\epsilon}_r & & \\ S^{2p(p^2-1)} & & & \end{array}$$

is commutative. \square

Recalling the main result of [2], we now define

$$\epsilon_r(t) \in \pi_*^{s_{2(p-1)[tp^2+(t-1)p+r]-2}}$$

where $r = 1, \dots, p - 1$, and $t > 0$, by the commutativity of the diagram

$$\begin{array}{ccc}
 S^{2tp(p^2-1)}V^{(p-r)}(1) & \xrightarrow{\Delta_t^r} & V^{(p-r)}(1) \\
 \uparrow \cup & & \downarrow c \\
 S^{2tp(p^2-1)} & \xrightarrow{\epsilon_r(t)} & S^{2(p-r)(p-1)+2}
 \end{array}$$

and obtain

THEOREM 3. *Let $p > 3$ be a prime. Then the elements*

$$\epsilon_r(t) \in \pi_{2(p-1)[tp^2+(t-1)p+r]-2}^s \begin{cases} r = 1, \dots, p - 1, \\ t > 0, \end{cases}$$

are all nonzero of order p . Moreover $\epsilon_r(1) = \epsilon_r$ and $\epsilon_r(p - 1) = \beta_{rp}$. Thus each element $\epsilon_1, \dots, \epsilon_{p-1}$ starts an infinite family in π_{*}^s . \square

Note that the mapping cones of Δ_t^r provide new examples of cyclic modules over Ω_{*}^U that are realizable.

Complete details will appear as part IV of the series [2].

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF VIRGINIA,
CHARLOTTESVILLE, VIRGINIA 22901