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A COMPLETE LOCAL FACTORIAL RING OF DIMENSION 4 WHICH IS NOT COHEN-MACAULAY

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Samuel [7] stated that he knew of no factorial noetherian ring which was not Cohen-Macaulay. Murthy [6] showed that a geometric factorial ring which is Cohen-Macaulay is Gorenstein. Subsequently, Bertin [1] constructed an example of a factorial ring which was not Cohen-Macaulay. Hochster and Roberts [5] noticed that such examples abound and were found by Serre [9]. On the other hand, Raynaud, Boutot, and Hartshorne and Ogus [3] have shown that a complete local ring which is factorial, of dimension at most 4, and with $\mathbf{C}$ as residue class field is Cohen-Macaulay.

This note is to announce that the completion of Bertin's example (which is characteristic 2) is factorial. This defeats a conjecture suggested by Example 5.9 of Hochster [4] which states: If A is a complete noetherian domain, then some symbolic power of a prime ideal of height one is a maximal Cohen-Macaulay module.

Let k be a perfect field of characteristic p with $p \neq 0$. Let N operate on k^4 by $N(e_i) = e_{i+1}$ for $1 \leqslant i < 4$ and with $N(e_4) = 0$. Then $I + N$ is an automorphism of k^4 of order p if $p \geqslant 5$ and of order 4 if $p = 2$. Let $B = k[X_1, X_2, X_3, X_4]$, which we consider to be the symmetric algebra on k^4. Let G denote the group of automorphisms of B induced by $I + N$. It follows from Samuel [8] that the ring of invariants $A = B^G$ is factorial. If $p = 2$, then Bertin [1] has shown that A is not Cohen-Macaulay. Using a result in Serre [9], Hochster and Roberts [5] show that A is not Cohen-Macaulay if $p \geqslant 5$. Let $S = B_m$ and let $R = S^G$, where $m = (X_1, X_2, X_3, X_4)$. It follows that $R = A_n$ with $n = m \cap A$. The different $D(S/R) = S$, and therefore the cohomology group $H^1(G, \mathbf{G}_m(S)) = 0$. Let $\hat{S}$ denote the m-adic completion of B. The first result is almost obvious.

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PROPOSITION 1. *The* $\mathfrak{n}$*-adic completion of* R *is the ring of* G*-invariants of* $\hat{S}$. *That is* $\hat{R} = \hat{S}^G$.

This yields the following corollary.

COROLLARY 2. *The ring* $\hat{R}$ *is not Cohen-Macaulay.*

Let $U_n = 1 + m^n S$ and $\hat{U}_n = 1 + m^n \hat{S}$. The U_n are subgroups of $\mathbf{G}_m(S)$, and the following sequences are exact as G-modules:

$$1 \longrightarrow U_1 \longrightarrow \mathbf{G}_m(S) \longrightarrow k^* \longrightarrow 1,$$

$$1 \longrightarrow U_{n+1} \longrightarrow U_n \longrightarrow m^n/m^{n+1} \longrightarrow 0$$

(and similarly with hats everywhere). Since G is cyclic, the cohomology of G is periodic of period 2 (cf. Cartan and Eilenberg [2]). We will study the exact sequence

$$\begin{aligned} \cdots \longrightarrow \hat{H}^0(G, m^n/m^{n+1}) \longrightarrow H^1(G, U_{n+1}) &\longrightarrow H^1(G, U_n) \\ &\longrightarrow H^1(G, m^n/m^{n+1}) \longrightarrow H^2(G, U_{n+1}) \\ &\longrightarrow H^2(G, U_n) \longrightarrow H^2(G, m^n/m^{n+1}) \longrightarrow \cdots \end{aligned}$$

and the corresponding one with hats. Note that the G-module m^n/m^{n+1} is the nth symmetric power of m/m^2 as a G-module.

PROPOSITION 3. *The connecting homomorphisms* $\hat{H}^0(G, m^n/m^{n+1}) \longrightarrow H^1(G, U_{n+1})$ *are zero for all* n. *Therefore the groups* $H^1(G, U_n)$ *are zero and the sequence*

$$\begin{aligned} 0 \longrightarrow H^1(G, \hat{U}_{n+1}) \longrightarrow H^1(G, \hat{U}_n) &\longrightarrow \cdots \\ \longrightarrow H^2(G, \hat{U}_n) \longrightarrow H^2(G, m^n/m^{n+1}) &\longrightarrow 0 \end{aligned}$$

is exact.

REMARK. The contragredient representation of G on the k-duals of m^n/m^{n+1} induces isomorphisms of k-vector spaces:

$$H^1(G, m^n/m^{n+1}) = H^2(G, (m^n/m^{n+1})\check{\ })\check{\ }.$$

To show that $\hat{R}$ is factorial, it is sufficient, therefore, to show that the homomorphisms $H^1(G, \hat{U}_n) \longrightarrow H^1(G, m^n/m^{n+1})$ are zero for all n. In characteristic $p = 2$, this is accomplished by directly calculating the groups $H^1(G, m^n/m^{n+1})$ and then showing that the connecting homomorphisms to $H^2(G, U_{n+1})$ are injections. Similar arguments should suffice in characteristic $p \geqslant 5$.

PROPOSITION 4. *Suppose* char $k = 2$. *If* n *is odd, then* $H^1(G, m^n/m^{n+1}) = 0$. *If* n *is even, then* $\dim_k H^1(G, m^n/m^{n+1}) = [n/4] + 1$. *If* $n = 4k$ *and* $x = X_1(X_1 + X_3)$ (*which is just* $X_1 \cdot (I + N)^2(X_1)$), *then a basis for* $H^1(G, m^n/m^{n+1})$ *is given by the classes of* $x^{2k}, x^{2k-1}a(x), \cdots, x^k a(x)^k$. *A basis for* $H^2(G, m^n/m^{n+1})$ *is given by the classes of* $(x + a(x))^{2k}, x^{2k-1}a(x) + xa(x)^{2k-1}, \cdots, x^k a(x)^k$, *where* $a(x) =: (I + N)(x)$ (*and similarly for* $n = 4k + 2$).

The results announced here, as well as similar ones for $\mathbf{Z}/p\mathbf{Z}$ acting on $k[[X_0, \cdots, X_{p-1}]]$, will appear elsewhere.

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