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UNITARY NILPOTENT GROUPS AND HERMITIAN K-THEORY. I

BY SYLVAIN E. CAPPELL[1]

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This announcement computes the Wall surgery obstruction groups of amalgamated free products of finitely presented groups by using the new UNil functors introduced below. Special cases of these results **[C4]** were obtained as consequences of the splitting theorems of **[C3]**. The present results use the general results on manifold decomposition outlined in **[C7]**. Further applications to the study of manifolds and submanifolds, Poincaré duality spaces, diffeomorphism groups, and Novikov's conjecture **[C8]** will be presented elsewhere.

1. UNil **of bimodules with involution.** Let R be a ring with unit and involution. Let M be an R-bimodule with involution; i.e. M is equipped with a homomorphism $x \to \bar{x}$ satisfying $\bar{\bar{x}}=x$, $(\alpha x \beta)^{-}=\bar{\beta}\bar{x}\bar{\alpha}$, $x \in M$, $\alpha, \beta \in R$. Call M hyperbolic if there is a decomposition of R-bimodules $M=N \oplus \bar{N}$, $\bar{N}=\{\bar{x} | x \in N \subset M\}$.

By a $(-1)^k$ Hermitian form over M we mean a triple (P, λ, μ) where P is a finitely generated free right R-module and $\lambda: P \times P \to M$, $\mu: P \to M/\{x-(-1)^k \bar{x} | x \in M\}$ satisfy:

(i) for $x \in P$ fixed, $y \to \lambda(x, y)$ is an R-homomorphism $P \to M$;
(ii) $\lambda(x, y)=(-1)^k(\lambda(y, x))^{-}$, $x, y \in P$;
(iii) $\lambda(x, x)=\mu(x)+(-1)^k(\mu(x))^{-}$ in M, $x \in P$;
(iv) $\mu(x+y)=\mu(x)+\mu(y)+\lambda(x, y)$, $x, y \in P$;
(v) $\mu(x\alpha)=\bar{\alpha}\mu(x)\alpha$, $x \in P$, $\alpha \in P$.

Let M_1 and M_2 be R-bimodules with involution which are free left R-modules. A (resp; simple) $(-1)^k$ UNil form over (M_1, M_2) is $C=(P_1, \lambda_1, \mu_1; P_2, \lambda_2, \mu_2)$ with $P_2=P_1^*$ and (P_i, λ_i, μ_i) a $(-1)^k$ Hermitian form over M_i, $i=1, 2$, for which there exist finite filtrations of R-modules

$$P_1 = P_1^0 \supset P_1^1 \supset P_1^2 \supset \cdots \supset P_1^n = 0,$$
$$P_2 = P_2^0 \supset P_2^1 \supset P_2^2 \supset \cdots \supset P_2^m = 0$$

so that, letting $\rho_1=P_1 \to P_2 \otimes_R M_1$ denote the adjoint of λ_1 and $\rho_2: P_2 \to P_1 \otimes_R M_2$ denote the adjoint of λ_2,

$$\rho_1(P_1^i) \subset P_2^{i+1} \otimes_R M_1, \qquad \rho_2(P_2^i) \subset P_1^{i+1} \otimes_R M_2, \qquad i \geqq 0$$

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(resp; and $(P_1, P_2; \rho_1, \rho_2)$ represents the zero element in the group of nilpotent objects $(\widetilde{\mathrm{Nil}}(R; M_1, M_2))$ defined in **[W1]**). Set $-C=(P_1, -\lambda_1, -\mu_1; P_2, -\lambda_2, -\mu_2)$. Call C a (resp; simple) kernel if there are free summands V_i of P_i, $i=1, 2$, with $V_2 \subset P_2 = P_1^*$ the annihilator of $V_1 \subset P_1$, and with $(\lambda_i | V_i \times V_i)$ and $(\mu_i | V_i)$ zero, $i=1, 2$ (resp; and also for $\rho_1' : P_1/V_1 \to P_2/V_2 \otimes_R M_1$, $\rho_2' : P_2/V_2 \to P_1/V_1 \otimes_R M_2$, the induced maps, $(P_1/V_1, P_2/V_2; \rho_1', \rho_2')$ represents zero in $(\widetilde{\mathrm{Nil}}(R; M_1, M_2))$). Note that $C \oplus (-C)$ is a (resp; simple) kernel.

Introduce among the (resp; simple) $(-1)^k$ UNil forms over (M_1, M_2) the equivalence relation generated by $A \sim B$ if $A \oplus (-B)$ is a (resp; simple) kernel. The equivalence classes form under the direct sum operation an abelian group denoted $\mathrm{UNil}^h_{2k}(R; M_1, M_2)$ (resp; $\mathrm{UNil}^s_{2k}(R; M_1, M_2)$). Give $R[t, t^{-1}]$ the involution $(xt^i)^- = \bar{x} t^{-i}$, $x \in R$, and similarly introduce involutions on $M_i \otimes_R R[t, t^{-1}]$. Now define

$$\begin{aligned}&\mathrm{UNil}^h_{2k-1}(R; M_1, M_2)\\ &\quad = \mathrm{UNil}^s_{2k}(R[t, t^{-1}]; M_1 \otimes_R R[t, t^{-1}], M_2 \otimes_R R[t, t^{-1}])/\mathrm{UNil}^s_{2k}(R; M_1, M_2).\end{aligned}$$

If R is a regular ring, or even just coherent of finite global homological dimension, define

$$\mathrm{UNil}^s_{2k-1}(R; M_1, M_2) = \mathrm{UNil}^h_{2k-1}(R; M_1, M_2).$$

Note the semiperiodicity $\mathrm{UNil}^x_n(R; M_1, M_2) \cong \mathrm{UNil}^x_{n+2}(R; M_1^-, M_2^-)$, $x=s$ or h, where M_i^- is M_i equipped with the involution $x \to -\bar{x}_i$.

2. **Surgery groups of free products with amalgamation.** Let $R \subset \Lambda_1$, $R \subset \Lambda_2$, be inclusions of rings with identity and involution. Assume Λ_i has an R-bimodule with involution decomposition $\Lambda_i = R \oplus \hat{\Lambda}_i$, $\hat{\Lambda}_i$ a free left R-module. A $(-1)^k$ UNil form $(P_1, \lambda_1, \mu_1; P_2, \lambda_2, \mu_2)$ over $(\hat{\Lambda}_1, \hat{\Lambda}_2)$ determines a $(-1)^k$ Hermitian form (P, λ, μ) over the free product with amalgamation ring $\Lambda_1 *_R \Lambda_2$ with $P = (P_1 \oplus P_2) \otimes_R (\Lambda_1 *_R \Lambda_2)$ and with,

$$\begin{aligned}\lambda(x, y) &= \langle x, y \rangle \quad \text{for } x \in P_2,\ y \in P_1 \text{ (recall } P_2 = P_1^*), \\ \lambda(x, y) &= \lambda_i(x, y) \quad \text{for } x, y \in P_i, \quad i = 1, 2,\end{aligned} \tag{1}$$

$$\mu(x) = \mu_i(x) \quad \text{for } x \in P_i, \quad i = 1, 2. \tag{2}$$

This construction induces for all n a homomorphism $\mathrm{UNil}^h_n(R; \hat{\Lambda}; \hat{\Lambda}_2) \to L^h_n(\Lambda_1 *_R \Lambda_2)$, the Wall surgery group of $\Lambda_1 *_R \Lambda_2$.

THEOREM 1. (i) *The image of* $\mathrm{UNil}^h_n(R; \hat{\Lambda}, \hat{\Lambda}_2) \to L^h_n(\Lambda_1 *_R \Lambda_2)$ *is 2-primary.*

(ii) *If* $\hat{\Lambda}_1$ *and* $\hat{\Lambda}_2$ *are hyperbolic, or if* 2 *is invertible in* R, *the image of* $\mathrm{UNil}^h_n(R; \hat{\Lambda}_1, \hat{\Lambda}_2)$ *in* $L^h_n(\Lambda_1 *_R \Lambda_2)$ *is zero.*

Theorem 1 is proved algebraically by adapting the proof of [**C3**, Lemma II.10] and of Remark 2 at the end of [**C3**, §2].

In the remainder of this paper, R is a ring with $Z \subset R \subset Q$. The groups H, G_1, G_2 are finitely presented, $H \subset G_1$ and $H \subset G_2$. Moreover, G_1 and G_2 are assumed equipped with homomorphisms $\omega_i: G_i \to Z_2 = \{\pm 1\}$, with $(\omega_1|H) = (\omega_2|H)$; as usual, these determine the involution on $R[G_i]$ with $\bar{g} = \omega(g)g^{-1}$, $g \in G_i \subset R[G_i]$, $i=1, 2$. Let $R[\hat{G}_i]$ denote the $R[H]$ subbimodule with involution of $R[G_i]$ additively generated by $g \in \{G_i - H\}$.

THEOREM 2. *The homomorphism* $\mathrm{UNil}_n^h(R[H]; R[\hat{G}_1], R[\hat{G}_2]) \to L_n^h(R[G_1 *_H G_2])$ *is a split monomorphism.*

The splitting ϕ of this homomorphism is defined as follows. Realize $x \in L_n^h(R[G_1 *_H G_2])$, using [**W2**] for $R=Z$ and [**CS**] for general $R \subset Q$, by a normal cobordism of 1_Y to $f: W \to Y$, Y a closed $(n-1)$-dimensional manifold, $n \geqq 6$, with $\pi_1(Y) = G_1 *_H G_2$, f an R-homotopy equivalence. Then from [**C7**], define $\phi(x)$ to be the splitting obstruction for f along $X \subset Y$, where $\pi_1 X = H$. Thus the action of $L_n^h(Z[G_1 *_H G_2])$ on $\mathscr{S}^h(Y)$, the set of h-cobordism classes of manifolds equipped with a homotopy equivalence to Y, restricts to a free action of $\mathrm{UNil}_n^h(Z[H]; Z[\hat{G}_1], Z[\hat{G}_2])$ on $\mathscr{S}^h(Y)$.

COROLLARY 3. $\mathrm{UNil}_n^h(R[H]; R[\hat{G}_1], R[\hat{G}_2])$ *is a 2-primary group. If* $\frac{1}{2} \in R$, *it is zero.*

Call a subgroup K of a group J square-root closed if $g^2 \in K$ implies $g \in K$ for $g \in J$ [**C3**]. For example, if K is normal in J, K is square-root closed in J if and only if J/K has no elements of order 2. Any subgroup of a finite group of odd order is square-root closed. If H is square-root closed in G_1, $Z[\hat{G}_1]$ is a hyperbolic $Z[H]$-bimodule with involution, hence:

COROLLARY 4. *If H is square-root closed in G_1 and G_2,*

$$\mathrm{UNil}_n^h(R[H]; R[\hat{G}_1], R[\hat{G}_2])$$

is zero.

Thus, many results of [**C3**] can be obtained from [**C7**] using Theorem 1(ii). From the general splitting obstruction theory of [**C7**] we get:

THEOREM 5. (i) *For Φ the quadrad of rings*

$$\begin{array}{ccc} R[H] & \to & R[G_1] \\ \downarrow & & \downarrow \\ R[G_2] & \to & R[G_1 *_H G_2] \end{array}$$

$$L_n^h(\Phi) = \mathrm{UNil}_n^h(R[H]; R[\hat{G}_1], R[\hat{G}_2]) \oplus H^{n-1}(Z_2; \mathrm{Ker}(K_0(R[H]) \to K_0(R[G_1]) \oplus K_0(R[G_2]))).$$

(ii) *Let*

$$\hat{L}_n^h(R[G_1 *_H G_2]) = \mathrm{CoKer}(\mathrm{UNil}_n^h(R[H]; R[\hat{G}_1], R[\hat{G}_2]) \to L_n^h(R[G_1 *_H G_2])).$$

Then if

$$H^i(Z_2; \mathrm{Ker}(K_0(R[H]) \to K_0(R[G_1]) \oplus K_0(R[G_2]))) = 0, \qquad i \geqq 1,$$

there is a long exact sequence

$$\cdots \to L_n^h(R[H]) \to L_n^h(R[G_1]) \oplus L_n^h(R[G_2]) \to \hat{L}_n^h(R[G_1 *_H G_2]) \to L_{n-1}^h(R[H]) \to \cdots$$

COROLLARY 6. *There is a long exact sequence for $x=h$ or for $x=s$,*

$$\cdots \to L_n^x(R[H]) \otimes Z[\tfrac{1}{2}] \to (L_n^x(R[G_1]) \oplus L_n^x(R[G_2])) \otimes Z[\tfrac{1}{2}] \to L_n^x(R[G_1 *_H G_2]) \otimes Z[\tfrac{1}{2}] \to L_{n-1}^x(R[H]) \otimes Z[\tfrac{1}{2}] \to \cdots$$

COROLLARY 7. *If*

$$H^i(Z_2; \mathrm{Ker}(K_0(R[H]) \to K_0(R[G_1]) \oplus K_0(R[G_2]))) = 0, \qquad i \geqq 1,$$

and if $\frac{1}{2} \in R$ or H square-root closed in G_1 and G_2, there is a long exact sequence

$$\cdots \to L_n^h(R[H]) \to L_n^h(R[G_1]) \oplus L_n^h(R[G_2]) \to L_n^h(R[G_1 *_H G_2]) \to L_{n-1}^h(R[H]) \to \cdots$$

Let $\mathscr{G}_0$ denote the smallest set of groups satisfying:

(i) $0 \in \mathscr{G}_0$;

(ii) if $H, G_1, G_2 \in \mathscr{G}_0$, with $H \subset G_i$, $i=1, 2$, then $G_1 *_H G_2 \in \mathscr{G}_0$;

(iii) if $H, J \in \mathscr{G}_0$ and $\xi_i : H \to J$, $i=1, 2$, are monomorphisms, then $J *_H \{t\} \in \mathscr{G}_0$, where

$$J *_H \{t\} = Z * J/\{t\xi_1(x)t^{-1}\xi_2(x)^{-1} \mid x \in H,\ t \text{ the generator of } Z\}.$$

From (iii), if $H \in \mathscr{G}_0$, then $Z \times H \in \mathscr{G}_0$. More generally, if $A, B \in \mathscr{G}_0$, then $A \times B \in \mathscr{G}_0$. $\mathscr{G}_0$ contains all torsion free finitely-generated one-relator groups and all fundamental groups of irreducible sufficiently large 3-manifolds.

Using **[C2]**, **[C4]**, a special case of the following result was proved in **[Q]**. From Corollary 3 we get

COROLLARY 8. *Let $L_n^s(G)$ denote the Wall surgery obstruction group for the simple homotopy equivalence problem for oriented manifolds with fundamental group G. If $G \in \mathscr{G}_0$,*

$$L_n^s(G) \otimes Z[\tfrac{1}{2}] \cong KO_n(K(G, 1)) \otimes Z[\tfrac{1}{2}]$$

and

$$L_n^s(G) \otimes Q \cong \bigoplus_{i \in Z} H_{n+4i}(G; Q).$$

This implies for a much larger set of groups than $\mathcal{G}_0$, Novikov's conjecture on homotopy invariance of the higher signatures [**C8**].

Problem. Let π be the group of a locally flat knot $S^1 \subset S^3$; does the abelianization homomorphism $\pi \to Z$ induce an isomorphism of Wall groups [**C1**]? The present results show that $L_n^s(\pi) = L_n^s(Z) \oplus$ (a 2-primary group). For π the group of a fibered knot, this 2-primary group is zero [**C4**].

3. **Applications to Wall groups of free products.** For $0 \leqq m_i \leqq \infty$, $i=0, \pm 1$, $R(m_{-1}, m_0, m_1)$ denotes the free R-module on generators x_i, y_j, z_k, $0 < i \leqq m_{-1}$, $0 < j \leqq 2m_0$, $0 < k \leqq m_1$, with involution determined by $\bar{x}_i = -x_i$, $\bar{z}_k = z_k$, $\bar{y}_{2j} = y_{2j-1}$. If G is a group with $\omega : G \to Z_2 = \{\pm 1\}$ determining the involution on $R[G]$, then $R[\hat{G}] \cong R(m_{-1}, m_0, m_1)$, where m_0 is $\frac{1}{2}$ the number of $g \in G$ with $g^2 \neq 1$, m_i is the number of $g \in G$ satisfying $g^2 = 1$, $g \neq 1$, $\omega(g) = i$, for $i = \pm 1$.

PROPOSITION 9. *For R a ring with $Z \subset R \subset Q$:*
(i)

$$\begin{aligned} \mathrm{UNil}_n^h(R; R(a,b,c), R(d,e,f)) &\cong \mathrm{UNil}_n^s(R; R(a,b,c), R(d,e,f)) \\ &\cong \mathrm{UNil}_{n+2}^h(R; R(c,b,a), R(f,e,d)) \end{aligned}$$

is 2-primary (2-torsion) for n odd (even).

(ii) *If $\frac{1}{2} \in R$, or n odd and $m_{-1} + m_1 + m'_{-1} + m'_1 = 0$, or $n = 2k$ and $m_{(-1)^{k+1}} + m'_{(-1)^{k+1}} = 0$, then*

$$\mathrm{UNil}_n^h(R; R(m_{-1}, m_0, m_1), R(m'_{-1}, m'_0, m'_1)) = 0.$$

(iii) *If $n = 2k$, $\frac{1}{2} \notin R$, $m_{(-1)^{k+1}} + m'_{(-1)^{k+1}} \neq 0$, $m_{-1} + m_0 + m_1 \neq 0$ and $m'_{-1} + m'_0 + m'_1 \neq 0$, then*

$$\mathrm{UNil}_n^h(R; R(m_{-1}, m_0, m_1), R(m'_{-1}, m'_0, m'_1)) \cong \bigoplus_{\infty} Z_2.$$

Let $\tilde{L}_n^s(G)$ denote the reduced surgery group, so that $L_n^s(G) = \tilde{L}_n^s(G) \oplus L_n(0)$. The following extends results of [**L**], [**C2**], [**C3**], [**C4**], [**C6**].

THEOREM 10. *Let G_1 and G_2 be finitely presented groups. Then*

$$L_n^s(G_1 * G_2) \cong L_n(0) \oplus \tilde{L}_n^s(G_1) \oplus \tilde{L}_n^s(G_2) \oplus A,$$

where A is

(i) *for $n=4k$, zero;*

(ii) *for $n=4k+1$ or $4k+3$, zero if G_1 and G_2 have no elements of order 2, and otherwise a 2-primary group,*

(iii) *for $n=4k+2$, zero if and only if $G_1=0$, or $G_2=0$ or G_1 and G_2 have no elements of order 2; otherwise it is a vector space over Z_2 of infinite rank.*

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DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON, NEW JERSEY 08540

INSTITUT DES HAUTES ÉTUDES SCIENTIFIQUES, PARIS, FRANCE

Current Address: Courant Institute of Mathematics, New York University, New York, New York 10012