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FINITE-DIMENSIONAL REPRESENTATIONS OF SEPARABLE C^*-ALGEBRAS

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Let $\mathscr{H}$ be a separable, infinite-dimensional, complex Hilbert space, and let $\mathscr{L}(\mathscr{H})$ denote the algebra of all bounded linear operators on $\mathscr{H}$. Furthermore, let $\mathscr{K}$ denote the (norm-closed) ideal of all compact operators in $\mathscr{L}(\mathscr{H})$, and let $\pi:\mathscr{L}(\mathscr{H})\to\mathscr{L}(\mathscr{H})/\mathscr{K}$ denote the canonical quotient map of $\mathscr{L}(\mathscr{H})$ onto the Calkin algebra. If T is any operator in $\mathscr{L}(\mathscr{H})$, we shall denote by $\mathscr{C}^*(T)$ the C^*-algebra generated by T and $1_{\mathscr{H}}$. Moreover, the C^*-algebra $\pi(\mathscr{C}^*(T))$, which is clearly the C^*-subalgebra of the Calkin algebra generated by $\pi(T)$ and 1, will be denoted by $\mathscr{C}_e^*(T)$. If $\mathscr{A}$ is any C^*-algebra, an *n-dimensional representation* of $\mathscr{A}$ is, by definition, a $*$-algebra homomorphism φ of $\mathscr{A}$ into the C^*-algebra $\boldsymbol{M}_n$ of all $n\times n$ complex matrices such that $\varphi(1)=1$. Such a representation φ will be called *irreducible* if $\varphi(\mathscr{A})=\boldsymbol{M}_n$.

The first objective of this note is to announce the following theorem, which gives, via the standard decomposition theory, a characterization of all finite-dimensional representations of a separable C^*-algebra. See [2].

THEOREM 1. *Let $\mathscr{A}$ be a separable C^*-subalgebra of $\mathscr{L}(\mathscr{H})$, and let φ be an irreducible n-dimensional representation of $\mathscr{A}$. Then, either*

(a) $\mathscr{A}\cap\mathscr{K}\subset$ kernel φ *(equivalently, there exists an n-dimensional representation $\tilde{\varphi}$ of the C^*-algebra $\pi(\mathscr{A})$ such that $\varphi(A)=\tilde{\varphi}(\pi(A))$ for every A in $\mathscr{A}$), in which case there exists a projection P in $\mathscr{L}(\mathscr{H})$ with infinite rank and nullity such that $\pi(P)$ commutes with the algebra $\pi(\mathscr{A})$, and there exists a $*$-algebra isomorphism ψ from the C^*-algebra $\pi(\mathscr{A})\pi(P)$ $(=\{\pi(A)\pi(P): A\in\mathscr{A}\})$ onto $\boldsymbol{M}_n$ such that $\varphi(A)=\psi(\pi(A)\pi(P))$ for every A in $\mathscr{A}$, or*

(b) $\mathscr{A}\cap\mathscr{K}\not\subset$ kernel φ, *in which case there exist a projection Q in $\mathscr{A}$ of finite rank that commutes with $\mathscr{A}$ and a $*$-algebra isomorphism η from the C^*-algebra $\mathscr{A}Q$ $(=\{AQ:A\in\mathscr{A}\})$ onto $\boldsymbol{M}_n$ such that $\varphi(A)=\eta(AQ)$ for every A in $\mathscr{A}$.*

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In view of this theorem and the fact that the n-dimensional representations of the C^*-algebras $\mathscr{C}^*(T)$ and $\mathscr{C}_e^*(T)$ are determined by their values at the points T and $\pi(T)$, respectively, the following definition is natural.

DEFINITION. Let T be an operator in $\mathscr{L}(\mathscr{H})$, and let n be a positive integer. Then the *reducing* $n\times n$ *spectrum of* T is the set $R^n(T)$ consisting of all those matrices L in $\boldsymbol{M}_n$ for which there exists an n-dimensional representation φ of $\mathscr{C}^*(T)$ such that $\varphi(T)=L$. Likewise, the *reducing* $n\times n$ *essential spectrum* of T is the set $R_e^n(T)$ consisting of all those matrices L in $\boldsymbol{M}_n$ for which there exists an n-dimensional representation ψ of $\mathscr{C}_e^*(T)$ such that $\psi(\pi(T))=L$.

Since for a fixed T in $\mathscr{L}(\mathscr{H})$, there is an obvious homeomorphism between $R^n(T)$ $[R_e^n(T)]$ and the topological space of all n-dimensional representations of $\mathscr{C}^*(T)$ $[\mathscr{C}_e^*(T)]$ (with the pointwise convergence topology), it is possible to study this set of representations by studying the set $R^n(T)$ $[R_e^n(T)]$. The second objective of this note is to announce some results of such a study. Other related results are simultaneously being announced in the *Notices*, since a new format requirement prevents their inclusion here. Proofs are given in [**2**].

It is easy to see that for every T in $\mathscr{L}(\mathscr{H})$, the sets $R^n(T)$ and $R_e^n(T)$ are compact, and $R_e^n(T)\subset R^n(T)$. Also, if $R^n(T)$ $[R_e^n(T)]$ is nonvoid, then $R^{kn}(T)$ $[R_e^{kn}(T)]$ is nonvoid for every positive integer k. Moreover, it turns out (Theorems 2 and 3) that $R^1(T)$ and $R_e^1(T)$ are the reducing spectrum and the reducing essential spectrum of T, respectively, as defined in [**1**].

In what follows, $\boldsymbol{C}_n$ will denote the n-dimensional Hilbert space of all (column) n-tuples of complex numbers, and elements of $\boldsymbol{M}_n$ will be regarded as operators on $\boldsymbol{C}_n$ via the obvious identification.

THEOREM 2. *Let T belong to $\mathscr{L}(\mathscr{H})$ and let n be any positive integer. If L belongs to $R^n(T)$ and either L is irreducible or L is a direct sum of irreducible matrices, no two of which are unitarily equivalent, then there exists a sequence $\{B_k\}_{k=1}^{\infty}$ of isometries $B_k:\boldsymbol{C}_n\to\mathscr{H}$ such that*

$$(*)\qquad \lim_{k\to\infty}(\|TB_kB_k^*-B_kLB_k^*\|+\|T^*B_kB_k^*-B_kL^*B_k^*\|)=0.$$

On the other hand, if L is any matrix in $\boldsymbol{M}_n$ and $\{B_k\}$ is a sequence of isometries $B_k:\boldsymbol{C}_n\to\mathscr{H}$ such that $(*)$ holds, then $L\in R^n(T)$.

THEOREM 3. *If T belongs to $\mathscr{L}(\mathscr{H})$, n is a positive integer, and $L=(\lambda_{ij})_{i,j=1}^n$ belongs to $\boldsymbol{M}_n$, the following conditions are equivalent*:

(a) $L\in R_e^n(T)$;

(b) *there exists a sequence $\{B_k\}_{k=1}^{\infty}$ of isometries $B_k:\boldsymbol{C}_n\to\mathscr{H}$ with mutually orthogonal ranges satisfying* $(*)$;

(c) *there exists a sequence $\{B_k\}_{k=1}^{\infty}$ of isometries $B_k:\boldsymbol{C}_n\to\mathscr{H}$ that converges weakly to* 0 *and satisfies* $(*)$;

(d) *there exist* n^2 *partial isometries* $\{W_{ij}\}_{i,j=1}^n$ *in* $\mathscr{L}(\mathscr{H})$ *satisfying the following conditions*:

(i) $W_{ij}W_{km}=\delta_{jk}W_{im}$, $1\leqq i, j, k, m\leqq n$;

(ii) $W_{ij}^*=W_{ji}$, $1\leqq i, j\leqq n$;

(iii) $Q=\sum W_{ii}$ *has infinite rank and nullity and* $QT-TQ\in\mathscr{K}$;

(iv) $W_{ii}TW_{jj}-\lambda_{ij}W_{ij}$ *is a trace class operator with arbitrarily small trace norm*, $1\leqq i, j\leqq n$.

THEOREM 4. *Let* T *belong to* $\mathscr{L}(\mathscr{H})$, *let* n *be any positive integer, and let* Ω *be any open set in* $\boldsymbol{M}_n$ *containing* $R^n(T)$ $[R_e^n(T)]$. *Then there exists a positive number* ε *such that if* $\|S-T\|<\varepsilon$, *then* $R^n(S)\subset\Omega$ $[R_e^n(S)\subset\Omega]$. *In particular, if* $R^n(T)$ $[R_e^n(T)]$ *is void, then* $R^n(S)$ $[R_e^n(S)]$ *is void for all* S *sufficiently close to* T *in the norm topology.*

THEOREM 5. *For each positive integer* n, *let* $\mathscr{R}_n$ *denote the set of all operators in* $\mathscr{L}(\mathscr{H})$ *having an* n*-dimensional reducing subspace. Then the norm closure of the set* $\bigcup_{k=1}^n\mathscr{R}_k$ *coincides with* $\{T\in\mathscr{L}(\mathscr{H}): R^k(T)\neq\varnothing$ *for some* $1\leqq k\leqq n\}$.

THEOREM 6. *Let* T *be an operator on* $\mathscr{H}$ *such that* $R^n(T)\neq\varnothing$ $[R_e^n(T)\neq\varnothing]$, *and let* $\mathscr{J}^n(T)$ $[\mathscr{J}_e^n(T)]$ *denote the closed ideal that is the intersection of the kernels of all* n*-dimensional representations of* $\mathscr{C}^*(T)$ $[\mathscr{C}_e^*(T)]$. *Then the* C^**-algebra* $\mathscr{C}^*(T)/\mathscr{J}^n(T)$ $[\mathscr{C}_e^*(T)/\mathscr{J}_e^n(T)]$ *is* $*$*-isomorphic to a subalgebra of the* C^**-algebra of all continuous functions from the compact set* $R^n(T)$ $[R_e^n(T)]$ *into* $\boldsymbol{M}_n$. *If* $n=1$, *the isomorphism is onto.*

BIBLIOGRAPHY

1. N. Salinas, *Reducing essential eigenvalues*, Duke Math. J. **40** (1973), 561–580.

2. C. Pearcy and N. Salinas, *Finite-dimensional representations of* C^**-algebras and the reducing matrical spectra of an operator* (to appear).

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