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SQUARE-INTEGRABLE KERNELS AND GROWTH ESTIMATES FOR THEIR SINGULAR VALUES

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Let $K(x, y)$, $0 \leqq x, y \leqq \pi$, be Lebesgue square-integrable. Define

$$K^{(r)}(x, y) \equiv \partial^r K(x, y)/\partial x^r \qquad (r = 0, 1, 2, \cdots, s)$$

for nonnegative integer s, and assume that $K(x, y)$ is extended, as an even function of x if s is even, and as an odd function of x if s is odd, into the domain $-\pi \leqq x \leqq 0$, and thence as a periodic function of x with period 2π. Let the singular values μ_n, where

$$\phi_n(x) = \mu_n \int_0^\pi K(x, y)\Psi_n(y)\, dy,$$

$$\Psi_n(x) = \mu_n \int_0^\pi \overline{K(y, x)}\phi_n(y)\, dy$$

with $\|\phi_n\|$, $\|\Psi_n\| \neq 0$, be ordered (indexed) in the natural manner according to increasing size, namely $0 < \mu_1 \leqq \mu_2 \leqq \mu_3 \leqq \cdots$.

In a perhaps overlooked paper, Smithies [**8**] has shown that

THEOREM 1. *If the $K^{(r)}(x, y)$, $0 \leqq r \leqq s-1$, are continuous in x, a.e. in y, and $K^{(s)}(x, y)$ is in $\mathscr{L}^p(x)$, a.e. in y, for some $1 < p \leqq 2$, then*

$$(1) \qquad \int_0^\pi \left[\int_0^\pi |K^{(s)}(x+h, y) - K^{(s)}(x-h, y)|^p\, dx \right]^{2/p} dy \leqq C\,|h|^{2\alpha}$$

for constant C, where $\alpha > 0$ if $s > 0$, $\alpha > (2-p)/2p$ if $s = 0$, implies

$$1/\mu_n = O(1)/n^{\alpha+s+1-1/p} \qquad \textit{as } n \to \infty.$$

Recognizing (1) as essentially an integrated Lipschitz condition, and using various properties associated with the class of kernels which satisfy such a condition, we can substantially generalize the above result.

We say that $K^{(s)}(x, y)$ is in Lip α if

$$|K^{(s)}(x+h, y) - K^{(s)}(x-h, y)| < |h|^\alpha A(y) \qquad (0 < \alpha \leqq 1)$$

where $A(y)$ is nonnegative and square-integrable. Likewise $K^{(s)}(x, y)$ is said to be (relatively uniformly) of bounded variation if for all $N \geqq 1$ and

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arbitrary choice of partition $0 \leqq x_0 \leqq x_1 \leqq \cdots \leqq x_N \leqq \pi$,

$$\sum_{n=1}^{N} |K^{(s)}(x_n, y) - K^{(s)}(x_{n-1}, y)| < B(y)$$

where $B(y)$ also is nonnegative and square-integrable. More generally, we say that $K^{(s)}(x, y)$ belongs to $\mathrm{Lip}(\alpha, p)$ if

$$\int_0^\pi |K^{(s)}(x + h, y) - K^{(s)}(x - h, y)|^p \, dx < |h|^{\alpha p} A^p(y) \qquad (0 < \alpha \leqq 1)$$

with $\mathscr{L}^2 A \geqq 0$.

Analogous to the one-variable situation (see Hardy and Littlewood [5], [6], for example), we may establish

Property 1. Kernels in $\mathrm{Lip}(\alpha, p)$ also belong to $\mathrm{Lip}(\alpha, q)$ for all $1 \leqq q < p$. Kernels in $\mathrm{Lip}\ \alpha$ are automatically in $\mathrm{Lip}(\alpha, p)$ for all $p \geqq 1$.

Property 2. Kernels which are (relatively uniformly) of bounded variation belong to $\mathrm{Lip}(1, 1)$.

Property 3. If $K(x, y)$ is absolutely continuous in x, a.e. in y, and

$$\int_0^\pi \left[\int_0^\pi |K^{(1)}(x, y)|^p \, dx \right]^{2/p} dy < \infty, \qquad p > 1,$$

then $K(x, y)$ is in $\mathrm{Lip}(1, p)$.

Property 4. If a kernel belongs both to $\mathrm{Lip}(\alpha, p)$ and to $\mathrm{Lip}(\beta, q)$ with $p < q$, then it belongs to $\mathrm{Lip}(\gamma, r)$ for all $p \leqq r \leqq q$, where

$$\gamma = \alpha \frac{p(q - r)}{r(q - p)} + \beta \frac{q(r - p)}{r(q - p)} .$$

Property 1 permits the immediate extension of Theorem 1 to the case $p > 2$ with the resultant growth estimate

$$1/\mu_n = O(1)/n^{\alpha+s+1/2} \qquad \text{as } n \to \infty.$$

In view of Property 3, moreover, we have essentially the generalization for arbitrary $p > 1$ of a result originally established for $p = 2$ by Chang [1].

One of the more general results which can be established using the "convexity" Property 4 is

THEOREM 2. *Let $K(x, y)$ satisfy the hypotheses of Theorem* 1 *for some nonnegative integer s and $p > 1$. If $K^{(s)}(x, y)$ belongs both to* $\mathrm{Lip}(\alpha, p)$ *and to* $\mathrm{Lip}(\beta, q)$, *with $p < q$, then $1/\mu_n = O(1)n^{-\sigma}$ as $n \to \infty$ where*

(i) *for $q \leqq 2$,*

$$\begin{aligned} \sigma &= \alpha + s + 1 - 1/p, & pq(\alpha - \beta) &> q - p, \\ &= \beta + s + 1 - 1/q, & pq(\alpha - \beta) &\leqq q - p, \end{aligned}$$

(ii) *for* $p \leqq 2 < q$,

$$\begin{aligned}
\sigma &= \alpha + s + 1 - 1/p, && pq(\alpha - \beta) > q - p, \\
&= \frac{q(2\beta + \alpha p + 2s + 1) - p(2\alpha + \beta q + 2s + 1)}{2(q - p)}, && 0 < pq(\alpha - \beta) \leqq q - p, \\
&= \beta + s + 1/2, && \alpha \leqq \beta,
\end{aligned}$$

(iii) *and for* $p > 2$,

$$\begin{aligned}
\sigma &= \alpha + s + 1/2, && \alpha > \beta, \\
&= \beta + s + 1/2, && \alpha \leqq \beta.
\end{aligned}$$

When $s=0$, *the additional provisos* $\alpha + 1/2 > 1/p$ *and* $\beta + 1/2 > 1/q$ *may be needed.*

In view of the several properties above, Theorem 2 contains as special cases the generalizations of other recent results of the author [**3**]. Furthermore, since they pertain to the singular values, any such conclusions must perforce also extend our knowledge concerning the growth behavior of the *characteristic* values of "smooth" kernels (see Hille and Tamarkin [**7**], Cochran [**2**], for example).

The relationship of the above theorems and their sundry special cases to summability results for the classical Fourier coefficients of integrable functions of a single real variable satisfying various smoothness criteria will be explored at length elsewhere [**4**].

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