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PERSISTENT MANIFOLDS ARE NORMALLY HYPERBOLIC

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Let M be a compact, smooth, boundaryless manifold, and let $\mathrm{Diff}^1(M)$ be the group of C^1-diffeomorphisms of M endowed with the C^1 topology. If $f \in \mathrm{Diff}^1(M)$, a compact invariant manifold of f, V, is a C^1 boundaryless submanifold of M, such that $f(V)=V$. In [**1**] and [**2**], M. Hirsch, C. Pugh and M. Shub proved that a normally hyperbolic compact invariant manifold (Definition 1 below) is persistent (Definition 2 below). The main purpose of this note is to announce the proof of the converse of this theorem.

1. The main theorem.

DEFINITION 1. If $f \in \mathrm{Diff}^1(M)$, a compact invariant manifold of f, V, is normally hyperbolic if the tangent bundle of M restricted to V, TM/V, has a splitting $TM/V=TV\oplus N^sV\oplus N^uV$ where TV is the tangent bundle of V, and N^sV, N^uV, are Tf-invariant subbundles of TM/V such that there exist constants $K>0$, $0<\lambda<1$ satisfying

$$\|(Tf)^n_x/(N^sV)_x\| \leqq K\lambda^n,$$

$$\|(Tf)^{-n}_x/(N^uV)_x\| \leqq K\lambda^n,$$

$$\|(Tf)^n_x/(N^sV)_x\| \; \|(Tf)^{-n}_{f^n(x)}/(TV)_{f^n(x)}\| \leqq K\lambda^n,$$

$$\|(Tf)^{-n}_x/(N^uV)_x\| \; \|(Tf)^n_{f^{-n}(x)}/(TV)_{f^{-n}(x)}\| \leqq K\lambda^n,$$

for all $x \in V$, $n \in Z^+$.

DEFINITION 2. If $f \in \mathrm{Diff}^1(M)$, a compact invariant manifold of f, V, is persistent if there exist a neighborhood $\mathscr{U}$ of f in $\mathrm{Diff}^1(M)$ and a neighborhood U of V in M such that:

(a) For all $g \in \mathscr{U}$, $V_g=\bigcap_{n\in Z} g^n(U)$ is a C^1 boundaryless submanifold of M and $V_f=V$.

(b) If g is C^1 near to f, V_g is C^1 near to $V_f=V$.

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[1] The results announced here are part of the author's doctoral thesis at IMPA under the guidance of J. Palis.

THEOREM. *A compact invariant manifold V of a diffeomorphism is persistent if and only if V is normally hyperbolic.*

As we observed above, the fact that normally hyperbolicity implies persistence is already known.

2. **Other results.** In the course of the proof of the main theorem the following situation arises frequently.

DEFINITION. Let K be a compact space, NK a vector bundle over K with a Finsler structure $\|\cdot\|$, and Nf an isomorphism of NK covering f. We say that K is Nf-isolated in NK if for all $x \in K$, $v \in (NK)_x$, $v \neq 0$, the set $\{\|(Nf)^n_x v\| \mid n \in \boldsymbol{Z}\}$ is not bounded. If $f \in \mathrm{Diff}^1(M)$, we say that it is a quasi-Anosov diffeomorphism if M is Tf-isolated in TM.

The following question is natural: Does quasi-Anosov imply Anosov? A partial answer is given by the following proposition.

THEOREM. *A quasi-Anosov diffeomorphism satisfies Axiom* A *and the no cycle condition. It is Anosov if and only if it is structurally stable, or, if and only if the dimensions of the stable manifolds of the periodic points are equal.*

The following result relates this problem to a situation studied by Hirsch in **[5]**.

PROPOSITION. *If f is a quasi-Anosov diffeomorphism of M, there exist a manifold N, a C^∞ embedding $i: M \to N$ and a C^1 diffeomorphism g of N such that $g \circ i = i \circ f$ and $i(M)$ is a hyperbolic set for g* **[4]**. *Moreover*:

$$\dim M = \dim N + \left(\max_{x \in \mathrm{Per}(f)} \dim W^u(x) - \min_{x \in \mathrm{Per}(f)} \dim W^s(x)\right),$$

where $W^s(x)$ and $W^u(x)$ are the stable and unstable manifolds of x.

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