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CONVEX MATRIX EQUATIONS

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1. **Introduction.** Let Δ_n denote the set of all $n \times n$ complex matrices A whose spectral norm $\|A\|$ is at most one. Then Δ_n forms a convex topological semigroup under matrix multiplication ([**6**], [**7**]). The subsemigroup Σ_n of Δ_n, consisting of all real nonnegative matrices in Δ_n, is the set of all $n \times n$ *doubly substochastic matrices*; that is, real nonnegative matrices whose row and column sums are at most one. The subsemigroup of Σ_n consisting of all $n \times n$ *doubly stochastic matrices* will be denoted by Ω_n.

Geometrically, Ω_n is the convex hull of the group of all $n \times n$ permutation matrices ([**1**], [**8**]), while Σ_n is the convex hull of the semigroup of all $n \times n$ subpermutation matrices [**9**]. The following theorem establishes a similar result for Δ_n.

THEOREM 1. *Δ_n is the convex hull of the set of all $n \times n$ unitary matrices.*

The proof of the theorem can be obtained by establishing that the unitary matrices form the set of extreme points of Δ_n. The result then follows by the Krein-Milman theorem. The complete proof will appear elsewhere [**10**]. Another proof of this result is given in [**15**].

Several authors have considered matrix equations involving doubly stochastic matrices. In particular, S. Sherman [**14**] and S. Schreiber [**13**] have considered the solvability of the equation $AX = B$ and D. J. Hartfiel [**5**] has considered the solvability of the equation $AXB = X$, where A, B, and X are doubly stochastic. The main purpose of this note is to consider the system of matrix equations

$$AX = B \quad \text{and} \quad BY = A, \tag{1.1}$$

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where A and B are arbitrary complex or real $m \times n$ matrices and where X and Y are in Δ_n, Σ_n or Ω_n. These ideas are then used in §§3 and 4 to investigate the Green's relations and regularity in those semigroups.

2. **The equations $AX = B$, $BY = A$.** The following theorems characterize the solvability of the equations (1.1) over Δ_n, Σ_n and Ω_n in terms of solvability over certain matrix groups.

THEOREM 2. *For arbitrary $m \times n$ complex matrices A and B, the equations* (1.1) *are solvable for $X, Y \in \Delta_n$ if and only if $A = BU$ for some unitary matrix U.*

THEOREM 3. *For arbitrary $m \times n$ real matrices A and B, the equations* (1.1) *are solvable for $X, Y \in \Sigma_n$ $[\Omega_n]$ if and only if $A = BP$ for some permutation matrix P.*

The proof of Theorem 2 is based on a theorem of Witt [**16**] and is fairly straightforward. Although it might be expected that the proof of Theorem 3 would follow from or be similar to the proof of Theorem 2, an entirely different approach is apparently needed. The proof for Σ_n and Ω_n is based in part on a theorem of Hardy, Littlewood and Pólya [**4**, Theorem 4.6]. These solvability theorems will now be used to investigate the algebraic structures of the convex semigroups Δ_n, Σ_n and Ω_n.

3. **The Green's relations**. The Green's relations $\mathscr{R}$, $\mathscr{L}$, $\mathscr{J}$, $\mathscr{H}$, and $\mathscr{D}$ play a fundamental role in the study of the algebraic structure of semigroups. For an arbitrary semigroups S with $a, b \in S$, the relation $\mathscr{R}$ $[\mathscr{L}, \mathscr{J}]$ is defined by $a \mathscr{R} b$ $[a \mathscr{L} b, a \mathscr{J} b]$ if and only if a and b generate the same principal right [left, two-sided] ideal in S. Then $\mathscr{H} = \mathscr{R} \cap \mathscr{L}$ and $\mathscr{D}$ is defined to be the join $\mathscr{R} \vee \mathscr{L}$ [**2**, Chapter 3]. The problem of characterizing the Green's relations on Δ_n, Σ_n and Ω_n can be solved by characterizing solutions to certain matrix equations. The results in this section follow from Theorems 2 and 3, together with their duals obtained by taking transposes in the equations (1.1). Notice that $\mathscr{D} = \mathscr{J}$ on these semigroups since they are compact [**6**].

THEOREM 4. *Let A and B belong to the semigroup Δ_n. Then*
(i) *$A \mathscr{R} B$ if and only if $A = BU$ for some unitary matrix U;*
(ii) *$A \mathscr{L} B$ if and only if $A = VB$ for some unitary matrix V;*
(iii) *$A \mathscr{H} B$ if and only if $A = BU = VB$ for some unitary matrices U, V;*
(iv) *$A \mathscr{D} B$ if and only if $A = VBU$ for some unitary matrices U and V.*

THEOREM 5. *Let A and B belong to the semigroup Σ_n of doubly substochastic matrices. Then*

(i) $A \mathscr{R} B$ *if and only if* $A = BP$ *for some permutation matrix* P;
(ii) $A \mathscr{L} B$ *if and only if* $A = QB$ *for some permutation matrix* Q;
(iii) $A \mathscr{H} B$ *if and only if* $A = BP = QB$ *for some permutation matrices* P *and* Q;
(iv) $A \mathscr{D} B$ *if and only if* $A = QBP$ *for some permutation matrices* P *and* Q.

It follows that Theorem 5 also characterizes the Green's relations on the semigroup Ω_n of doubly stochastic matrices [**11**], since the permutation matrices are contained in Ω_n. Moreover, these characterizations can be used to show that the maximal subgroups of Δ_n are isomorphic to full unitary groups, while Σ_n and Ω_n have finitely many maximal subgroups, each of which is isomorphic to a direct product of full symmetric groups [**3**].

4. **Regularity**. An element a in a semigroup S is said to be *regular* if the equation $a = axa$ is solvable for $x \in S$. If in addition $x = xax$, then a and x are said to be *semi-inverses.* In this section the regular elements in Δ_n, Σ_n and Ω_n are investigated. Clearly not every matrix in Δ_n is regular. In particular, the only nonsingular regular members of Δ_n are the unitary matrices. The following concepts will facilitate the characterizations of regularity.

The *singular values* of an $n \times n$ complex matrix A are the positive square roots of the eigenvalues of A^*A, where A^* denotes the conjugate transpose of A. Now A is a *partial isometry* if the linear transformation represented by A preserves distances on the range of A^*. The *Moore-Penrose generalized inverse* [**12**] of A is the unique matrix A^+ defined by $A^+y = x$ if $Ax = y$ and x is in the range of A^*, and $A^+y = 0$ if $A^*y = 0$.

Now it can be shown [**10**] that $E \in \Delta_n$ is idempotent if and only if there is a unitary matrix U such that UEU^* is diagonal with 0's and 1's on the diagonal. This fact, together with the results in §3, lead to the following characterizations of regularity. The details of the proof will appear elsewhere ([**10**], [**11**]).

THEOREM 6. *Let* $A \in \Delta_n$ $[\Sigma_n, \Omega_n]$. *Then the following statements are equivalent.*

(i) A *is regular in* Δ_n $[\Sigma_n, \Omega_n]$.
(ii) A^* *is the unique semi-inverse of* A *in* Δ_n $[\Sigma_n, \Omega_n]$.
(iii) *The singular values of* A *are* 0 *and* 1.
(iv) A *is a partial isometry.*
(v) $\|A\| = 1$ *if* A *is nonzero.*
(vi) $A^+ = A^*$.

REFERENCES

1. G. Birkhoff, *Tres observaciones sobre el algebra lineal*, Univ. Nac. Tucumán Revista. Ser. A **5** (1946), 147–151.

2. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*. Vol. I, Math. Surveys, no. 7, Amer. Math. Soc., Providence, R. I., 1961. MR **24** #A2627.

3. H. K. Farahat, *The semigroup of doubly-stochastic matrices*, Proc. Glasgow Math. Assoc. **7** (1966), 178–183. MR **34** #2590.

4. G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, Cambridge Univ. Press, New York, 1934.

5. D. J. Hartfiel, *The matrix equation* $AXB = X$, Pacific J. Math. **36** (1971), 659–669.

6. K. H. Hofmann and P. S. Mostert, *Elements of compact semigroups*, Merrill, Columbus, Ohio, 1966.

7. A. S. Householder, *The theory of matrices in numerical analysis*, Blaisdell, Waltham, Mass., 1964. MR **30** #5475.

8. L. Mirsky, *Results and problems in the theory of doubly-stochastic matrices*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **1** (1962/63), 319–334. MR **27** #3007.

9. ———, *On a convex set of matrices*, Arch. Math. **10** (1959), 88–92. MR **21** #5643.

10. J. S. Montague, *Equations in a matrix semigroup* (submitted).

11. J. S. Montague and R. J. Plemmons, *Doubly stochastic matrix equations* (submitted).

12. R. Penrose, *A generalized inverse for matrices*, Proc. Cambridge Philos. Soc. **51** (1955), 406–413. MR **16**, 1082.

13. S. Schreiber, *On a result of S. Sherman concerning doubly stochastic matrices*, Proc. Amer. Math. Soc. **9** (1958), 350–353. MR **20** #1687.

14. S. Sherman, *A correction to "On a conjecture concerning doubly stochastic matrices"*, Proc. Amer. Math. Soc. **5** (1954), 998–999. MR **16**, 326.

15. J. Stoer, *On the characterization of least upper bound norms in matrix space*, Numer. Math. **6** (1964), 302–314.

16. E. Witt, *Theorie der quadratische Formen in beliebigen Korpern*, J. Reine Angew. Math. **176** (1937), 31–44.

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