BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 78, Number 5, September 1972

ABSOLUTELY TORSION-FREE RINGS

BY ROBERT A. RUBIN

Communicated by Oscar Goldman, March 6, 1972

Call a ring Λ *absolutely torsion-free* (ATF) if for every finite kernel functor σ, i.e. a topologizing filter of nonzero left ideals, $\sigma(\Lambda) = 0$. In this article we note the basic properties of ATF rings, and give some related results about hereditary noetherian prime rings. The notation and terminology to be used are that of Goldman [**1**]. In particular, if Λ is a ring, $K(\Lambda)$ (respectively $I(\Lambda)$) will denote the set of kernel functors (respectively idempotent kernel functors) of Λ.

1. **Absolutely torsion-free rings.** We first note that ATF rings are a generalization of the familiar concept of integral domain.

PROPOSITION 1.1. *Let R be a commutative ring. Then R is an integral domain if and only if for every $\sigma \in K(R)$, $\sigma \neq \infty \Rightarrow \sigma(R) = 0$.*

The partial ordering on $K(\Lambda)$ provides a useful description of ATF rings.

PROPOSITION 1.2. *Λ is ATF if and only if there is $\mu \in I(\Lambda)$, $\mu \neq \infty$, such that for all $\infty \neq \sigma \in K(\Lambda)$, $\sigma \leqq \mu$.*

A definition is required in order to free the concept of ATF ring from that of kernel functor. A submodule M of a module N is called a *weakly essential* submodule if for any finite subset $x_1, \ldots, x_n$ of N, there is $0 \neq r \in \Lambda$ such that $rx_i \in M$ for each i. We then have

THEOREM 1.3. *Λ is ATF if and only if every weakly essential left ideal of Λ is a rational left ideal.*

We note some elementary properties of ATF rings.

PROPOSITION 1.4. *If Λ is ATF then Λ is a prime ring, i.e. the product of nonzero ideals of Λ is nonzero. Furthermore, if R is the center of Λ, then R is an integral domain and Λ is a torsion-free R-module.*

The converse of the first part of the preceding proposition is false. For if k is a field, V an infinite-dimensional vector space over k, and

---

AMS 1970 *subject classifications.* Primary 16A08; Secondary 16A14.
Key words and phrases. Kernel functor, hereditary, prime ring, equivalence of categories, maximal ring of quotients, simple ring, von Neumann regular, selfinjective, Goldie ring, epimorphism of rings, flat module.

$\Lambda = \mathrm{Hom}_k(V, V)$, then Λ is more than just a prime ring; it is in fact primitive. But Λ is not ATF.

Under certain circumstances however, prime rings are ATF. The first is that the ring be "almost" commutative.

PROPOSITION 1.5. *Let Λ be a ring which is finitely generated as a module over its center R. Then Λ is ATF if and only if Λ is a prime ring.*

The next instance is somewhat more interesting.

THEOREM 1.6. *Let Λ be a left Goldie ring. Then Λ is ATF if and only if Λ is a prime ring.*

Some examples of ATF rings are the following:

(a) If Λ has no zero divisors, then Λ is ATF.

(b) If Λ is a simple ring, then Λ is ATF.

(c) If Λ satisfies the d.c.c. on either side, then Λ is ATF if and only if Λ is simple.

(d) Let k be a field, and let Λ be the polynomial ring over k in two noncommuting indeterminates, x and y, subject to the relations $x^2 = y^2 = 0$. Then Λ is noetherian and prime, so by Theorem 1.6 is ATF.

We now turn to the question of obtaining new ATF rings from a given one. The next theorem leads to an important method.

THEOREM 1.7. *Let Λ and Γ be rings, and suppose that the category of (left) Λ-modules is equivalent to the category of (left) Γ-modules. Then $K(\Lambda) \approx K(\Gamma)$ as partially ordered sets.*

COROLLARY 1.8. *If Λ is ATF, and $\Gamma = M_n(\Lambda)$, the $n \times n$ matrix ring with entries in Λ, then Γ is ATF.*

The formation of polynomial rings preserves our condition.

THEOREM 1.9. *Let Λ be ATF, and let $\Gamma = \Lambda[x]$, i.e. we adjoin to Λ a central nonzero-divisor, nonunit x. Then Γ is ATF.*

This theorem is proved by considering the connection between a left ideal of Γ and the left ideal of Λ consisting of its "leading coefficients."

Any ring between an ATF ring and its maximal ring of quotients (see Utumi [3]) is also ATF.

THEOREM 1.10. *Let Λ be an ATF ring, and let Q be its maximal ring of (left) quotients. If Γ is a ring such that $\Lambda \subseteq \Gamma \subseteq Q$, then Γ is ATF.*

Rather unexpectedly perhaps, there is a manner in which ATF rings fail to arise, namely from taking subrings. Let Λ be the ring of 2×2 lower triangular matrices over a field k. Then the maximal ring of quotients of Λ is $M_2(k)$, the full 2×2 matrix ring. But $M_2(k)$ is ATF while Λ, which has a nilpotent ideal, is not.

2. **The maximal ring of quotients of an** ATF **ring.**

THEOREM 2.1. *Let Λ be an ATF ring, and Q its maximal left ring of quotients. Then Q is simple, von Neumann regular, and selfinjective. Furthermore Q satisfies the d.c.c. if and only if Λ is (Goldie) finite dimensional as a left Λ-module.*

The only part of this theorem that is not well known is the simplicity of Q in the absence of any finiteness condition. It is shown that for any $0 \neq I$, a two-sided ideal of Q, Q is contained in a finite direct sum of copies of I. The selfinjectivity of Q then yields that $Q = I$.

3. **Hereditary noetherian prime rings.** We wish to show that every ring between an hereditary noetherian prime ring (HNP) and its maximal ring of quotients is itself HNP and a ring of quotients of the given HNP ring. We first note the following theorem.

THEOREM 3.1. *Let $f: \Lambda \to \Gamma$ be an epimorphism of rings. If f induces on Γ the structure of a flat right Λ-module, then there is a $\mu \in I(\Lambda)$ such that $\Gamma \approx Q_\mu(\Lambda)$.*

THEOREM 3.2. *Let Λ be an HNP ring and Q its maximal ring of quotients. Let Γ be a ring such that $\Lambda \subseteq \Gamma \subseteq Q$. Then there is a $\mu \in I(\Lambda)$ such that $\Gamma \approx Q_\mu(\Lambda)$.*

The crux of the proof of this theorem is Silver's characterization of an epimorphism of rings (see [2, Proposition 1.1]).

THEOREM 3.3. *Let $\sigma \in I(\Lambda)$, and suppose $\Lambda \subset Q_\sigma(\Lambda)$; i.e. $\sigma(\Lambda) = 0$. Let $\mathfrak{x}$ be a left ideal of $Q_\sigma(\Lambda)$, and let $\mathfrak{a} = \Lambda \cap \mathfrak{x}$. Then if $\mathfrak{a}$ is a finitely generated projective Λ-module, $\mathfrak{x}$ is a finitely generated projective $Q_\sigma(\Lambda)$-module.*

Here the result follows from the fact that $Q_\sigma(\Lambda)$ is σ-injective and that extensions are unique.

The preceding theorems and various results from §1 yield

THEOREM 3.4. *Let Λ be an HNP ring, with maximal ring of quotients Q. If Γ is a ring with $\Lambda \subseteq \Gamma \subseteq Q$, then Γ is HNP.*

REFERENCES

1. O. Goldman, *Rings and modules of quotients*, J. Algebra **13** (1969), 10–47. MR **39** #6914.

2. L. Silver, *Noncommutative localization and applications*, J. Algebra **7** (1967), 44–76. MR **36** #205.

3. Y. Utumi, *On quotient rings*, Osaka Math. J. **8** (1956), 1–18. MR **18**, 7.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KENTUCKY, LEXINGTON, KENTUCKY 40506