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A RADIAL AVERAGING TRANSFORMATION, CAPACITY AND CONFORMAL RADIUS

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Introduction. Let $\mathscr{D} = \{D_1, \ldots, D_n\}$ be a family of domains in the plane, containing the origin. We define a radial averaging transformation $\mathscr{R}_A$ on $\mathscr{D}$ by which we obtain a starlike domain D^*. When $\mathscr{D}$ is such that the domains $D_1, \ldots, D_n$ are obtained from a fixed domain D by rotation or reflexion, $\mathscr{R}_A$ becomes a radial symmetrization. One of the results we present is an inequality relating the conformal radius of D^* to the conformal radii of $D_1, \ldots, D_n$ at the origin. This result includes, as particular cases, the radial symmetrization results of Szegö [**6**] (for starlike domains), Marcus [**4**] (for general domains) and Aharonov and Kirwan [**1**]. The inequality for the conformal radii is obtained via an inequality for conformal capacities. A number of applications in the theory of functions is described.

1. Let M be the half strip $\{(x, y)|0 < x < 1, 0 < y\}$. We shall say that a function f is of class $\overline{B}(M)$ if

(i) f is continuous in $\overline{M}$ (= closure of M);

(ii) $0 \leqq f \leqq 1$ in M;

(iii) the set $\Omega_1 = \{(x, y)|f(x, y) < 1\} \cap M$ is bounded;

(iv) on any half line $\{x = x_0\} \cap \overline{M}, f$ assumes every value $\lambda, 0 < \lambda < 1$, at least once, but not more than a finite number of times;

(v) $f \in C^1(\overline{\Omega}(f))$, where $\Omega(f) = \{(x, y)|0 < f(x, y) < 1\} \cap M$;

(vi) for any line $x = x_0$, $0 \leqq x_0 \leqq 1$, the set of points on $\{x = x_0\} \cap \overline{\Omega}(f)$ where $\partial f/\partial y = 0$ is finite.

If $f \in \overline{B}(M)$ we denote

$$(1.1) \qquad \begin{aligned} \Omega_\lambda(f) &= \{(x, y)|f(x, y) < \lambda\} \cap M \qquad (0 < \lambda \leqq 1), \\ \Omega_0(f) &= \{(x, y)|f(x, y) = 0\} \cap M. \end{aligned}$$

$$(1.2) \qquad l(x_0, \lambda; f) = \operatorname{meas}(\{x = x_0\} \cap \Omega_\lambda(f)) \qquad (0 \leqq \lambda \leqq 1),$$

where the measure is the linear Lebesgue measure. We note that $l(x_0, \lambda; f)$ is a strictly monotonic increasing function of λ, $0 \leqq \lambda \leqq 1$.

We now introduce

DEFINITION 1.1. Let $\mathscr{F} = \{f_1, \ldots, f_n\} \subset \overline{B}(M)$ and let $A = \{a_1, \ldots, a_n\}$ be a set of positive numbers such that $\sum_{j=1}^n a_j = 1$. Denote

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$$(1.3) \qquad l^*(x, \lambda) = \sum_{j=1}^{n} a_j l(x, \lambda; f_j);$$

$$\Omega_\lambda^* = \Omega_\lambda^*(\mathscr{F}, A) = \{(x, y) | 0 < y < l^*(x, \lambda)\} \cap M \qquad (0 < \lambda \leqq 1),$$
$$(1.4) \quad \Omega_0^* = \Omega_0^*(\mathscr{F}, A) = \{(x, y) | 0 \leqq y \leqq l^*(x, 0)\} \cap M,$$
$$\Omega^* = \Omega^*(\mathscr{F}; A) = \Omega_1^* - \Omega_0^*.$$

Then the *linear averaging transformation* $\mathscr{L}_A$ on $\mathscr{F}$ is defined as follows:

$$(1.5) \qquad \begin{aligned} f^*(x, y) = \mathscr{L}_A(\mathscr{F}) &= 0, \quad \text{if } (x, y) \in \Omega_0^*, \\ &= \lambda, \quad \text{if } y = l^*(x, \lambda), 0 < \lambda < 1, \\ &= 1, \quad \text{if } (x, y) \in M - \Omega_1^*. \end{aligned}$$

The following two results are the main steps in the derivation of the main theorems.

LEMMA 1.1. *Let $\mathscr{F}$ and A be as in Definition* 1.1. *Then f^* is uniformly Lipschitz in M.*

THEOREM 1.1. *Let $\mathscr{F}$ and A be as in Definition* 1.1. *Let $G(t)$ be a function defined for $t \geqq 0$ such that $G(t)$ is continuous, convex and nondecreasing. Then, with the notations introduced above, we have*

$$(1.6) \quad \iint_{\Omega^*} G((1 + |\nabla f^*|^2)^{1/2})\, dx\, dy \leqq \sum_{j=1}^{n} a_j \iint_{\Omega(f_j)} G((1 + |\nabla f_j|^2)^{1/2})\, dx\, dy,$$

where $\Omega(f_j) = \Omega_1(f_j) - \Omega_0(f_j)$.

COROLLARY.

$$(1.7) \qquad \iint_{\Omega^*} |\nabla f^*|^p\, dx\, dy \leqq \sum_{j=1}^{n} a_j \iint_{\Omega(f_j)} |\nabla f_j|^p\, dx\, dy \qquad (1 \leqq p).$$

Note that the left side of (1.6) is meaningful because of Lemma 1.1.

2. A condenser in the plane is a system $C = (\Omega, E_0, E_1)$ where Ω is a domain and E_0, E_1 are disjoint closed sets whose union is the complement of Ω. We shall assume also that E_0 is compact and E_1 unbounded. An alternative notation for C will be $C = (D, E_0)$ where $D = \Omega \cup E_0$.

If Ω satisfies the segment property (i.e., for any point P on the boundary of Ω there exists a segment $\overline{PP'}$ lying outside Ω), there exists a unique function ω, called the *potential function* of C, such that ω is harmonic in Ω and continuous in the extended plane and such that $\omega \equiv 0$ on E_0 and $\omega \equiv 1$ on E_1. In this case the *conformal capacity* of C may be defined by

$$I(C) = \mathrm{Dir}_\Omega[\omega] \equiv \iint_\Omega |\nabla\omega|^2 \, dx\, dy. \tag{2.1}$$

We now introduce

DEFINITION 2.1. Let $\mathscr{D} = \{D_1, \dots, D_n\}$ be a family of open sets in the complex plane z. Suppose that the closed disk $|z - z_0| \leqq \rho$ (for some $\rho > 0$) is contained in $\bigcap_{j=1}^n D_j$. Let

$$K_j^\rho(\phi) = \{r | z = z_0 + re^{i\phi} \in D_j, \rho < r < \infty\} \qquad (0 \leqq \phi < 2\pi); \tag{2.2}$$

$$l_j^\rho(\phi) = \int_{K_j^\rho(\phi)} \frac{dr}{r} \quad \text{and} \quad R_j(\phi) \equiv R(\phi; D_j; z_0) = \rho \exp l_j^\rho(\phi). \tag{2.3}$$

(Note that $R_j(\phi)$ does not depend on ρ.)

Let $A = \{a_1, \dots, a_n\}$ be a set of positive numbers such that $\sum_{j=1}^n a_j = 1$. Set

$$R^*(\phi) = \prod_{j=1}^n R_j(\phi)^{a_j}; \tag{2.4}$$

$$D^* = \mathscr{R}_A(\mathscr{D}; z_0) = \{z = z_0 + re^{i\phi} | 0 \leqq r < R^*(\phi), 0 \leqq \phi < 2\pi\}. \tag{2.5}$$

We shall say that $\mathscr{R}_A$ is a *radial averaging transformation* on $\mathscr{D}$ with center z_0.

If $\{C_j\}_{j=1}^n$ is a family of condensers, $C_j = (\Omega_j, E_{0,j}, E_{1,j}) = (D_j, E_{0,j})$ where $\bigcap_1^n E_{0,j} \supseteq \{|z - z_0| \geqq \rho\}$ we define

$$C^* = \mathscr{R}_A(\{C_j\}; z_0) = (D^*, E_0^*) \tag{2.6}$$

where $D^* = \mathscr{R}_A(\{D_j\}; z_0)$ and $E_0^* = \mathscr{R}_A(\{E_{0,j}\}; z_0)$. ($E_0^*$ is defined in the same way as D^* except that in (2.5) we have $0 \leqq r \leqq R^*(\phi)$.)

We can now formulate the main result.

THEOREM 2.1. *Let $\{C_1, \dots, C_n\}$ be a family of condensers as above, and let C^* be defined as in* (2.6). *Suppose that the domains $\Omega_1, \dots, \Omega_n$ have the segment property. Then*

$$I(C^*) \leqq \sum_1^n a_j I(C_j). \tag{2.7}$$

The proof is based on Theorem 1.1. We may assume that $z_0 = 0$ and $\rho = 1$. We map the domain $|z| < 1$, cut along the positive real axis, by $w = \ln z$ onto the half strip $0 < u < \infty$, $0 < v < 2\pi$ ($w = u + iv$). Let ω_j be the potential function of C_j. Denote by $f_j(u, v)$ the function ω_j represented in (u, v) coordinates. Then we apply Theorem 1.1 (or, more precisely, inequality (1.7) with $p = 2$) to $\mathscr{F} = \{f_1, \dots, f_n\}$ in the strip mentioned above.

If D is a domain in the plane and $z_0 \in D$, denote by $r(z_0, D)$ the conformal (or inner) radius of D at z_0. (For definition and properties see for instance Hayman [**3**, pp. 78–83].) Using a theorem of Pólya and Szegö [**5**] on the relation between conformal radius and conformal capacity and Theorem 2.1 we obtain

THEOREM 2.2. *Let $\mathscr{D} = \{D_1, \ldots, D_n\}$ be a family of domains such that $z_0 \in \bigcap_1^n D_j$. Let $D^* = \mathscr{R}_A(\mathscr{D}; z_0)$ (Definition 2.1). Then*

$$\prod_{j=1}^{n} r(z_0, D_j)^{a_j} \leqq r(z_0, D^*). \tag{2.8}$$

As a first application of Theorem 2.2 we obtain the following symmetrization result:

THEOREM 2.3. *Let $f(z) = a_1 z + a_2 z^2 + \cdots$ be an analytic function in the unit disk $|z| < 1$. Let D be the image of $|z| < 1$ by $w = f(z)$. Let $A = \{a_j\}_1^n$ be a set of positive numbers such that $\sum_1^n a_j = 1$, let $\{\alpha_j\}_1^n$ be a set of integers ($\alpha_j \neq 0$) and let $\{\beta_j\}_1^n$ be an arbitrary set of real numbers.*

If $R(\phi) = R(\phi; D; 0)$ (see (2.3)) set

$$R^*(\phi) = \prod_{j=1}^{n} R(\alpha_j \phi + \beta_j)^{b_j}, \quad \textit{where } b_j = a_j/|\alpha_j|; \tag{2.9}$$

$$D^* = \{w = \sigma e^{i\phi} | 0 \leqq \sigma < R^*(\phi), 0 \leqq \phi < 2\pi\}. \tag{2.10}$$

Then

$$|a_1| \leqq r(0, D) \leqq r(0, D^*)^{1/b}, \quad \textit{where } b = \sum_1^n b_j. \tag{2.11}$$

Theorem 2.3 includes as particular cases the radial symmetrization results of Szegö [**6**], Marcus [**4**] and Aharonov and Kirwan [**1**].

We bring now two applications of the preceding theorems.

THEOREM 2.4. *Let $f(z)$ and D be as in Theorem 2.3. Denote*

$$D_t = \{w = \sigma e^{i\phi} | 0 \leqq \sigma < R(\phi)^t, 0 \leqq \phi < 2\pi\} \qquad (0 < t < 1), \tag{2.12}$$

where $R(\phi) = R(\phi; D; 0)$. Then

$$|a_1| \leqq r(0, D) \leqq r(0, D_t)^{1/t}. \tag{2.13}$$

THEOREM 2.5. *Let $f(z) = z + a_2 z^2 + \cdots$ and D be as in Theorem 2.3. Let $R^*(\phi)$ be defined as in (2.9). Suppose that $R^*(\phi) \leqq M \leqq \infty$, $0 \leqq \phi < 2\pi$. Suppose also that for some set of m rays issuing from the origin, with arguments $\phi_1, \ldots, \phi_m$ we have*

$$\sup_{1 \leqq j \leqq m} R^*(\phi_j) = K.$$

Let D_0 be the disk $|w| < M$ (the entire plane if $M = \infty$) cut along the rays $w = \sigma e^{i\phi_j}$, $K_0 \leqq \sigma < M$, $j = 1, \ldots, m$, where K_0 is so chosen that $r(0, D_0) = 1$. (It follows from our assumptions that $M \geqq 1$.) Then $K_0 \leqq K$.

Theorem 2.5 implies a number of special "covering" theorems such as Theorem 5 and 6 of Marcus [**4**] and Theorem 4.2 of Aharonov and Kirwan [**1**].

A complete presentation of the results described in this note and additional applications will appear elsewhere. We mention also that a discussion of radial averaging transformations with metrics of the form $g(r)\, dr\, d\phi$ is given in [**2**].

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