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GEOMETRIC THEORY OF DIFFERENTIAL EQUATIONS. THE LJAPUNOV INTEGRAL FOR MONOTONE COEFFICIENTS

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An equation

$$x'' + p(t)x = 0, \qquad -\infty < t < \infty, p(t) > 0, \tag{1}$$

can be considered as the Frenet equation of a locally convex curve $x(t) = (x_1(t), x_2(t))$ in unimodular centroaffine differential geometry [1]. The osculating ellipse E_u at $x(u)$ is the solution of

$$y'' + p(u)y = 0, \quad y(u) = x(u), \quad y'(u) = x'(u).$$

We prove an unimodular centroaffine Kneser theorem:

THEOREM 1. *If $p(t)$ is strictly monotone and differentiable in an interval $[a, b]$, then every osculating ellipse E_t, $t \in [a, b]$, contains all osculating ellipses of smaller area defined on the same interval in its interior.*

The area of the osculating ellipse is proportional to $p(t)^{-1/2}$. By the Jordan curve theorem, the assertion is true if it is true for neighboring points. Then it is easily checked that a pair of conjugate diameters of the smaller ellipse is in the interior of the larger one. The approximation and convergence theorems of convexity imply:

THEOREM 2. *If $p(t)$ is monotone and continuous in $[a, b]$, then every osculating ellipse E_t, $t \in [a, b]$, contains all osculating ellipses of smaller area defined on the same interval.*

The parameter $t-u$ is equal to two times the area covered by the radius vector of $x(t)$ and $\int_u^t p(\tau)\, d\tau$ is equal to two times the area covered by the radius vector of the polar reciprocal $x^*(t)$ of x for the unit circle, if $x(t)$ is a curve of unit Wronskian [1, §3]. For p monotone increasing, the curve x and the osculating ellipses E_t $(t \geqq u)$ are contained in E_u, and $x^*(\tau)$, E_τ^* $(u \leqq \tau \leqq t)$ are in E_t^*.

Let $\phi(u)$ be the conjugate point of u for (1), i.e., the zero following

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u of a nontrivial solution of (1) that vanishes at u. Let $\psi(u)$ be the co-conjugate point of u for (1), i.e., the zero following u of the derivative of a nontrivial solution of (1) whose derivative vanishes at u. A major topic in the study of equations (1) are estimates of the Ljapunov integral

$$L(u) = [\phi(u) - u] \int_u^{\phi(u)} p(t)\, dt.$$

We suppose that $p(t)$ is monotone increasing and continuous and that $\phi(u) < \infty$. As an application of Theorem 2, we have

$$\pi p(\phi(u))^{-1/2} \leqq \phi(u) - u \leqq \pi p(u)^{-1/2},$$
$$\pi p(u)^{1/2} \leqq \int_u^{\phi(u)} p(t)\, dt.$$

If $\psi(u) \geqq \phi(u)$, then

$$\int_u^{\phi(u)} p(t)\, dt \leqq \pi p(\phi(u))^{1/2}.$$

If $\psi(u) < \phi(u)$, then

$$\int_u^{\phi(u)} p(t)\, dt \leqq \frac{3}{2} \pi p(\phi(u))^{1/2}.$$

(The difference $\phi(u) - \psi(u)$ has been investigated in [2].) Together, we obtain:

THEOREM 3. *For monotone increasing, continuous, positive* $p(t)$, *the Ljapunov integral satisfies*

$$\pi^2 \left(\frac{p(u)}{p(\phi(u))} \right)^{1/2} \leqq L(u) \leqq \left[1 + \frac{1+\epsilon}{4} \right] \pi^2 \left(\frac{p(\phi(u))}{p(u)} \right)^{1/2},$$
$$\epsilon = \operatorname{sgn}[\phi(u) - \psi(u)].$$

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